Asymptotic Solution of Stochastic Fluid Model with Upward Jumps

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ABSTRACT

This paper is interested in studying a type of production models-stocks that can be seen as a stochastic fluid flow system with upward jumps at level zero. The joint distribution of the stocks level and the controlling Markov process is governed by two differential systems with specific boundary conditions. The uniqueness of the solution of this problem has been proved. Also, a unified solution with no distinction between singular or invertible drift matrix is proposed. This method is based on the randomization technique, which is acknowledged by its numerical stability and accuracy.

Key Words: Markov process, Partial differential equations, Stochastic fluid models.

INTRODUCTION

Markov modulated fluid flow processes are a popular subject in applied probability. They are used in several applications especially in telecommunication networks modeling and production-inventory systems. In this paper, we study the fluid level distribution in a buffer of infinite capacity where the input and output rates of the fluid in the system are controlled by a finite homogeneous Markov process. When the fluid level hits zero, the quantity of fluid instantaneously jumps to a predetermined positive level. These kinds of models are used generally in production-inventory models. The fluid process represents the dynamic inventory level and the modulated Markov environment describes the production and sales seasons. In practice, each jump in the buffer content represents an instantaneous stock replenishment following an order arrival. This jump occurs when there is no commodity to deliver in order to avoid unsatisfied commands or lost sales.

Asmussen and Kella (2000) developed a general theory on a multi-dimensional martingale for Markov additive processes and apply it on a Brownian inventory model where upward jumps occur according to a phase-type process. The inventory level distribution has been determined via its Laplace transform. Miyazawa and Takada (2002) gave an exponential matrix form of the stationary distribution of a buffer content with downward jumps, where matrices are numerical solutions of an integral equation. Kulkarni and Yan (2007) studied inventory models with instant stock replenishments. They derive a system of first order non-homogeneous linear differential equations for the steady-state distribution. The solution is based on an eigenvalues/eigenvectors computation of a key matrix. They also studied the economic order quantity policy that minimizes the long-run average cost. Kulkarni and Yan (2012) expanded the model to ale low backlogging and exponential lead times. Matrix-analytic methods have been largely explored in stochastic fluid models. Da Silva and Latouche (2005) showed that the fluid queue is independent from the modulating process if the latter switches instantaneously to a state with a positive rate when the fluid level drops to zero. Latouche and Taylor (2009) give the probability density for models allowing changes of phase permitting sojourn in level zero. Barron *et al*. (2014) considered a production-inventory control model with two reflecting boundaries, representing the finite storage capacity and the finite maximum backlog. They combined the matrix-analytic approach and martingale notion to derive a closed form of various functional costs.

It is well-known that spectral methods are numerically instable, for more details see for instance Abbessi and Nabli (2008). For matrix-analytic methods, the solution is commonly expressed by means of an exponential matrix that depends on a key matrix, solution of a matrix equation. The resolution of this matrix equation to within a tolerance error may have an impact on the accuracy on the distribution of the fluid level. More recently, Nabli *et al*. (2016) showed among others that matrix-analytic methods are also numerically instable. In this paper, we deal with the asymptotic distribution of fluid level with a fixed upward jump at level zero. Kulkarni and Yan (2007) showed that the joint distribution of the fluid level and the modulating Markov process is governed by a system of first order non-homogeneous linear differential equations with specific boundary conditions. Our aim is to prove the uniqueness of the solution of differential system and to propose a numerical solution, which is neither spectral nor based on matrix-analytic method. It utilizes the uniformization technique which is acknowledged by its numerical stability since it involves only positive numbers bounded by one.

MATERIALS AND METHODS

We consider a production-inventory stochastic stock model, where the production and demand are supposed to be controlled by an irreducible Markov process $X=(X_t)_{t\geq 0}$ over a finite state space S. The infinitesimal generator of X is denoted by $A=(a_{ij})_{i,j\in S}$. When the modulated process X is in state i, the demand occurs at a constant rate c_i and the production is done at rate r_i. Thus, $d_i = r_i - c_i$ represents the effective input rate associated to state i: while the process X stays in state i, the fluid level increases or decreases at rate d_i depending on the sign of d_i. If d_i is null, the fluid level remains unchanged. When the system becomes empty, the fluid level jumps instantaneously to a fixed level $q>0$. Let us consider the following notations:

$$S_+ = \{i\in S \,/\, d_i > 0\},\ \mathrm{S}_- = \{i\in S \,/\, d_i < 0\}$$
$$\text{and } S_0 = \{i\in S \,/\, d_i = 0\}.$$

Let $\boldsymbol{\pi} = (\pi_i, i\in S)$ be the stationary probability distribution of $(X_t)_{t\geq 0}$:

$$\pi_i = \lim_{t\to\infty}\mathbb{P}(X_t = i) = \mathbb{P}(X_\infty = i).$$

The random variable X_∞ stands for the limit in distribution of $(X_t)_{t\geq 0}$. It is well-known that $\boldsymbol{\pi}$ satisfies the linear system of equations $\boldsymbol{\pi}A=\mathbf{0}$ and $\sum_{i\in S}\pi_i = 1$, where $\mathbf{0}$ is the null vector of dimension the cardinality of S. Let Q_t be the fluid level in the system at time t. The system is stable if and only if:

$$\mu := \sum_{i\in S} d_i \pi_i < 0. \qquad (1)$$

In this case, the limiting distribution of the process $(Q_t)_{t\geq 0}$, denoted Q_∞, exists. Let $F_i(x) = \mathbb{P}(Q_\infty > x, X_\infty = i)$ be the joint distribution of (Q_∞, X_∞), the probability $F_i(x)$ is equal to the limit of $\mathbb{P}(Q_t > x, X_t = i)$ when t tends to infinity. Kulkarni and Yan (2007) had shown that the row vector $\boldsymbol{F}(x) = (F_i(x), i\in S)$ satisfies the following differential equation:

$$(\mathcal{P})\begin{cases} \boldsymbol{F}'(x)D = \boldsymbol{F}(x)A + \boldsymbol{F}'(0)D, \forall 0^+ \leq x \leq q^- \\ \boldsymbol{F}'(x)D = \boldsymbol{F}(x)\mathrm{A}, & \forall x \geq q^+ \\ F_i(q^+) = F_i(q^-), & \forall i\in S\setminus S_0 \\ F_i'(0) = 0, & \forall i \in S_+ \\ \lim_{x\to\infty} F_i(x) = 0, & \forall i\in S \\ \sum_{i\in S} F_i(0) = 1, \end{cases}$$

where D is the diagonal matrix composed by the effective input rates d_i and $\boldsymbol{F}'(x)$ is the derivative of $\boldsymbol{F}(x)$ with respect to the variable x. Kulkarni (2007) had proved that, under the stability condition, the limiting distribution $\boldsymbol{F}(x)$ is continuous on $[0,\infty)$ and piecewise differentiable on $(0,q)$ and (q,∞).

RESULTS AND DISCUSSION

Our aim in this section is to prove the solution uniqueness of the problem $(\mathcal{P})$ assuming that the stability condition holds. The proof will be achieved in two steps. First, we suppose that the drift matrix D is invertible or equivalently $S_0 = \varnothing$. In this case, the functions $F_i(x); i \in S;$ are continuous at point q. Next, it will be proved that the case where the matrix D is singular comes down to the first case.

Theorem 2.1

If there is no vanishing effective input rates and under the stability condition (1), the problem $(\mathcal{P})$ has a unique solution.

Proof: Since the matrix D is nonsingular, the differential system on $(0,q)$ becomes $\boldsymbol{F}'(x) = \boldsymbol{F}(x)M + \boldsymbol{F}'(0)$, where $M = AD^{-1}$. First, we show that the vector space $\mathcal{C}$ of solutions of this differential system is of dimension $N+1$: $\dim \mathcal{C} = N+1$, where $\mathcal{C} = \{\boldsymbol{F} : (0,q) \to \mathbf{R}^N \text{ differentiable} / \boldsymbol{F}'(x) = \boldsymbol{F}(x)M + \boldsymbol{F}'(0)\}$.

For this purpose, let us consider the following correspondence:

$$\begin{aligned} \phi : \mathbf{R}\times\mathbf{R}^N &\to \mathcal{C} \\ (\alpha,\boldsymbol{b}) &\mapsto \phi(\alpha,\boldsymbol{b}) = \boldsymbol{F}_{\alpha,\boldsymbol{b}} \end{aligned}$$

where $\boldsymbol{F}_{\alpha,\boldsymbol{b}}$ is the element of $\mathcal{C}$ that satisfies $\sum_{i\in s} F_i(0) = \alpha$ and $\boldsymbol{F}'(0) = \boldsymbol{b}$. The correspondence ϕ is actually an application: each element $(\alpha,\boldsymbol{b})$ has one and only one image $\phi(\alpha,\boldsymbol{b})$. One can check easily that $\boldsymbol{b}\sum_{n\ge 1}\frac{x^n}{n!}M^{n-1} + \alpha\boldsymbol{\pi}$ is an element of $\mathcal{C}$ which satisfies the boundary conditions $\sum_{i\in s} F_i(0) = \alpha$ and $\boldsymbol{F}'(0) = \boldsymbol{b}$. The image $\phi(\alpha,\boldsymbol{b})$ is unique, otherwise there exist two elements $\boldsymbol{F}$ and $\boldsymbol{G}$ in $\mathcal{C}$ that fulfill the same boundary conditions. The difference $\boldsymbol{H} = \boldsymbol{F} - \boldsymbol{G}$, which is naturally in $\mathcal{C}$, satisfies $\sum_{i\in s} H_i(0) = 0$ and $\boldsymbol{H}'(0) = \mathbf{0}$. At point $x = 0^+$, the differential system gives $\boldsymbol{H}(0)M = \mathbf{0}$. Under the stability condition, Nabli and Ouerghi (2009) had proved that 0 is a simple eigenvalue of M. Since $\boldsymbol{\pi}M = \mathbf{0}$, then $\boldsymbol{H}(0) = \gamma\boldsymbol{\pi}$ for a real γ. The equality $\sum_{i\in s} H_i(0) = 0$ leads to state that $\gamma = 0$ and therefore $\boldsymbol{H}(0) = \mathbf{0}$. Taking into account of the condition $\boldsymbol{H}'(\text{x}) = \boldsymbol{H}(x)M$ and the last result $\boldsymbol{H}(0) = \mathbf{0}$, the only solution of the differential system is $\boldsymbol{H}(x) = \boldsymbol{H}(0)\exp(xM) = \mathbf{0}$. This proves the required result $\boldsymbol{F} = \boldsymbol{G}$. So, ϕ is an application which is clearly linear. The same arguments above prove that ϕ is injective and surjective, thus $\dim\mathcal{C} = \dim \mathbf{R}\times\mathbf{R}^N = N+1$. In other words, knowing the values of $\sum_{i\in s} F_i(0) = \alpha$ and $\boldsymbol{F}'(0) = \boldsymbol{b}$, the solution $\boldsymbol{F}$ on the interval $(0,q)$ is:

$$\boldsymbol{F}(x) = \boldsymbol{b}\sum_{n\ge 1}\frac{x^n}{n!}M^{n-1} + \alpha\boldsymbol{\pi}.$$

Now, for the second differential system related to the case $x > q$, the solution is $\boldsymbol{F}(x) = \boldsymbol{F}(q)\exp((x-q)M)$. The remaining boundary condition $\lim_{x\to\infty} F_i(x) = 0, \forall i \in S$ implies necessarily that $\boldsymbol{F}(q)$ is orthogonal to all eigenvectors associated to nonnegative eigenvalues of the matrix M. According to our previous analysis, the continuity at point $x = q$ and the equality $\sum_{i\in s} F_i(0) = 1$ permit to conclude that $\boldsymbol{F}(q) = \boldsymbol{F}'(0)\sum_{n\ge 1}\frac{q^n}{n!}M^{n-1} + \boldsymbol{\pi}.$ Nabli and Ouerghi (2009) proved that M has exactly $|S_-|$ nonnegative eigenvectors, where $|S_-|$ stands for the cardinality of S_-. Since $\boldsymbol{F}(q)$ is expressed by means of $\boldsymbol{F}'(0)$, the last result combined with the boundary condition $F_i'(0) = 0, \forall i \in S_+$ lead to define the vector $\boldsymbol{F}'(0)$ by a unique way. In conclusion, the solution of Problem $(\mathcal{P})$ is unique and it is equal to:

$$\boldsymbol{F}(x) = \begin{cases} \boldsymbol{F}'(0)\sum_{n\ge 1}\frac{x^n}{n!}M^{n-1} + \boldsymbol{\pi}, & \text{for } x \in [0,q] \\ \boldsymbol{F}(q)\exp((x-q)M), & \text{for } x \ge q \end{cases} \quad (2)$$

■

Now, let us deal with the general case where the drift matrix D is singular: $S_0 \ne \varnothing$.

Theorem 2.2

Under the stability condition (1), the solution of Problem $(\mathcal{P})$ exists and it is unique.

Proof: First of all, it is interesting to remark that the last condition in $(\mathcal{P})$ can be replaced by $\boldsymbol{F}(0) = \boldsymbol{\pi}$, which is a stronger condition. The infinitesimal generator A and the diagonal matrix D can be ordered according to the partition $S = \hat{S} \cup S_0$, where $\hat{S} = S_+ \cup S_-$:

$$A = \begin{pmatrix} A_{\hat{S}\hat{S}} & A_{\hat{S}S_0} \\ A_{S_0\hat{S}} & A_{S_0S_0} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} D_{\hat{S}} & 0_{\hat{S}S_0} \\ 0_{S_0\hat{S}} & 0_{S_0} \end{pmatrix},$$

ditto for the row vector $\boldsymbol{F}(x) = \left(\boldsymbol{F}_{\hat{S}}(x) \quad \boldsymbol{F}_{S_0}(x)\right)$. The two differential systems corresponding to the subset S_0 lead to the following shared system:

$$\boldsymbol{F}_{\hat{S}}(x)A_{\hat{S}S_0} + \boldsymbol{F}_{S_0}(x)A_{S_0S_0} = 0_{S_0}, \forall x > 0.$$

Since A is irreducible, the submatrix $A_{S_0S_0}$ is invertible (see for instance Bhat, 1984). The above system is equivalent to:

$$\boldsymbol{F}_{S_0}(x) = \boldsymbol{F}_{\hat{S}}(x)A_{\hat{S}S_0}(-A_{S_0S_0})^{-1}. \quad (3)$$

So, the determination of $\boldsymbol{F}_{\hat{S}}(x)$ in a unique way gives $\boldsymbol{F}_{S_0}(x)$ and therefore the whole solution $\boldsymbol{F}(x) = \left(\boldsymbol{F}_{\hat{S}}(x) \quad \boldsymbol{F}_{S_0}(x)\right)$ in a unique way. For this purpose, let us consider the two differential systems related to $\hat{S}$:

$$\begin{cases} \boldsymbol{F}_{\hat{S}}'(x)D_{\hat{S}} = \boldsymbol{F}_{\hat{S}}(x)A_{\hat{S}\hat{S}} + \boldsymbol{F}_{S_0}(x)A_{S_0\hat{S}} + \boldsymbol{F}_{\hat{S}}'(0)D_{\hat{S}}, \forall 0^+ \le x \le q^- \\ \boldsymbol{F}_{\hat{S}}'(x)D_{\hat{S}} = \boldsymbol{F}_{\hat{S}}(x)A_{\hat{S}\hat{S}} + \boldsymbol{F}_{S_0}(x)A_{S_0\hat{S}}, \quad \forall x \ge q^+ \end{cases}$$

By injecting Identity (3), the above differential systems turn into:

$$\begin{cases} \boldsymbol{F}_{\hat{S}}'(x)D_{\hat{S}} = \boldsymbol{F}_{\hat{S}}(x)\hat{A} + \boldsymbol{F}_{\hat{S}}'(0)D_{\hat{S}}, \forall 0^+ \le x \le q^- \\ \boldsymbol{F}_{\hat{S}}'(x)D_{\hat{S}} = \boldsymbol{F}_{\hat{S}}(x)\hat{A}, \quad \forall x \ge q^+ \end{cases}$$

where $\hat{A} = A_{\hat{S}\hat{S}} + A_{\hat{S}S_0}(-A_{S_0S_0})^{-1}A_{S_0\hat{S}}$. It is well known that $\hat{A}$ is an irreducible infinitesimal generator and that its steady-state vector, denoted by $\hat{\boldsymbol{\pi}}$, is:

$$\hat{\boldsymbol{\pi}} = \frac{\boldsymbol{\pi}_{\hat{S}}}{\sum_{i\in\hat{S}}\pi_i} = \frac{\boldsymbol{\pi}_{\hat{S}}}{1-\sum_{i\in S_0}\pi_i}, \text{ where } \boldsymbol{\pi}_{\hat{S}} = \left(\pi_i, i \in \hat{S}\right).$$

The summary of Problem $(\mathcal{P})$ on the subset $\hat{S}$ is:

$$(\hat{\mathcal{P}})\begin{cases} \boldsymbol{F}_{\hat{S}}'(x)D_{\hat{S}} = \boldsymbol{F}_{\hat{S}}(x)\hat{A} + \boldsymbol{F}_{\hat{S}}'(0)D_{\hat{S}}, \forall 0^+ \le x \le q^- \\ \boldsymbol{F}_{\hat{S}}'(x)D_{\hat{S}} = \boldsymbol{F}_{\hat{S}}(x)\hat{A}, & \forall x \ge q^+ \\ F_i(q^+) = F_i(q^-), & \forall i \in \hat{S} \\ F_i'(0) = 0, & \forall i \in S_+ = \hat{S}_+ \\ \lim_{x\to\infty} F_i(x) = 0, & \forall i \in \hat{S} \\ \boldsymbol{F}_{\hat{S}}(0) = \boldsymbol{\pi}_{\hat{S}} = \alpha\hat{\boldsymbol{\pi}}, \text{ where } \alpha = 1 - \sum_{i\in S_0}\pi_i \end{cases}$$

The stability condition associated to this problem is $\sum_{i\in\hat{S}} d_i\hat{\pi}_i < 0$, which is indeed satisfied since $\sum_{i\in S} d_i\pi_i = \alpha\sum_{i\in\hat{S}} d_i\hat{\pi}_i$. The hypothesis of Theorem 2.1 is actually satisfied for $(\hat{\mathcal{P}})$.

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It is useful to state that even if the matrix is singular, the solution of Problem $(\mathcal{P})$ remains continue at point q giving that $\boldsymbol{F}_{\hat{S}}$ is continue and $\boldsymbol{F}_{S_0}$ is a linear combination of the components of $\boldsymbol{F}_{\hat{S}}$. The property of continuity is crucial in the resolution of partial differential system related to stochastic fluid models. Some mathematical mistakes or incompleteness due basically to inattentiveness to a discontinuity property are reported in Nabli and Alwan (2016).

CLOSED FORM OF SOLUTIONS

Kulkarni and Yan (2007) expressed the solution by means of eigenvalues end eigenvectors of the matrix . Identity (2) gives an explicit solution depending of the matrix M. By contrast with Matrix-Analytic-Methods (MAM in short), Identity (2) does not involve any matrix that is solution of a matrix equation. In the next section, another closed form expression of the solution is proposed, it is neither spectral nor MAM. Moreover, it deals simultaneously with the case D invertible and D singular. Finally, it is useful to point out that despite the uniqueness of the solution related to Problem $(\mathcal{P})$, the approaches of resolution are multiple and therefore the expressions of the solution are multiple.

Simple case

This subsection deals with the case where the cardinality of the state space S is composed only by two states, one in S_+ and the other in S_-, denoted respectively by 1 and 0. The infinitesimal generator A and the diagonal drift matrix D are written as follows:

$$A = \begin{pmatrix} -\beta & \beta \\ \alpha & -\alpha \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} d_1 & 0 \\ 0 & d_0 \end{pmatrix}.$$

The next theorem gives a closed form for the joint distribution of the bivariate (Q_∞, X_∞).

Theorem 3.1

For the particular case $|S| = 2,$ *the solution of Problem* $(\mathcal{P})$ *is:*

$$\boldsymbol{F}(x) = \begin{cases} (1-\frac{x}{q})\boldsymbol{\pi} + \pi_1 \frac{e^{\gamma x}-1}{\gamma q}\begin{pmatrix} 1 & -\frac{d_1}{d_0} \end{pmatrix}, & \text{for } x \in [0, q) \\ \pi_1 \frac{e^{\gamma x} - e^{\gamma(x-q)}}{\gamma q}\begin{pmatrix} 1 & -\frac{d_1}{d_0} \end{pmatrix}, & \text{for } x \geq q, \end{cases}$$

where $\gamma = -(\frac{\beta}{d_1} + \frac{\alpha}{d_0}) = -\frac{(\alpha+\beta)\mu}{d_1 d_0}$ *and* $\boldsymbol{\pi} = \begin{pmatrix} \pi_1 = \frac{\alpha}{\alpha+\beta} & \pi_0 = \frac{\beta}{\alpha+\beta} \end{pmatrix}$ *is the stationary distribution of the process* $(X_t)_{t\geq 0}$.

Proof: It is easy to check that 0 and γ are the two eigenvalues of the matrix $M = AD^{-1}$. The associated eigenvectors are $\boldsymbol{d} = \begin{pmatrix} d_1 \\ d_0 \end{pmatrix}$ and $\begin{pmatrix} -\pi_0 \\ \pi_1 \end{pmatrix}$ respectively. So, it comes that:

$$M = AD^{-1} = P\begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix}P^{-1}, \text{ where } P = \begin{pmatrix} d_1 & -\pi_0 \\ d_0 & \pi_1 \end{pmatrix}.$$

The fourth boundary condition in $(\mathcal{P})$ gives $F_1^{'}(0) = 0$. The expression in Identity (2) for the particular value $x = q$ is equivalent to:

$$\begin{aligned} \boldsymbol{F}(q) &= \begin{pmatrix} 0 & F_0^{'}(0) \end{pmatrix} \sum_{n\geq 1} \frac{q^n}{n!} M^{n-1} + \boldsymbol{\pi} \\ &= \begin{pmatrix} 0 & F_0^{'}(0) \end{pmatrix} P \begin{pmatrix} q & 0 \\ 0 & \frac{e^{\gamma q}-1}{\gamma} \end{pmatrix} P^{-1} + \boldsymbol{\pi} \end{aligned}$$

It has been established in the poof of Theorem 2.1 that $\boldsymbol{F}(q)$ is orthogonal to all eigenvectors associated to nonnegative eigenvalues. In the case $|S| = 2,$ this property reduces to $\boldsymbol{F}(q) \perp \boldsymbol{d}$:

$$\begin{aligned} \boldsymbol{F}(q) \perp \boldsymbol{d} \quad &\Rightarrow \quad \begin{pmatrix} 0 & F_0^{'}(0) \end{pmatrix} P \begin{pmatrix} q & 0 \\ 0 & \frac{e^{\gamma q}-1}{\gamma} \end{pmatrix} P^{-1}\boldsymbol{d} + \boldsymbol{\pi d} = 0 \\ &\Rightarrow \quad q\begin{pmatrix} 0 & F_0^{'}(0) \end{pmatrix}\begin{pmatrix} d_1 \\ d_0 \end{pmatrix} + \boldsymbol{\pi d} = 0, \text{ since } P^{-1}\boldsymbol{d} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\Rightarrow \quad F_0^{'}(0) = -\frac{\mu}{q d_0}, \text{ since } \boldsymbol{\pi}\mathbf{d} = \mu. \end{aligned}$$

By replacing $F_0^{'}(0)$ by its value in the expression of $\boldsymbol{F}(q)$ mentioned above, it is easy to show that:

$$\boldsymbol{F}(q) = \pi_1 \frac{e^{\gamma q}-1}{\gamma q}\begin{pmatrix} 1 & -\frac{d_1}{d_0}\end{pmatrix}. \quad (4)$$

Let us deal now with the case $x \in [0,q)$. Thanks to Identity (2) and the expression of $F_0^{'}(0)$ established previously, we obtain:

$$\begin{aligned}\boldsymbol{F}(\mathrm{x}) &= \begin{pmatrix} 0 & -\frac{\mu}{qd_0}\end{pmatrix}\sum_{n\ge1}\frac{x^n}{n!}M^{n-1}+\boldsymbol{\pi}\\ &= \begin{pmatrix} 0 & -\frac{\mu}{qd_0}\end{pmatrix}P\begin{pmatrix} x & 0\\ 0 & \frac{e^{\gamma x}-1}{\gamma}\end{pmatrix}P^{-1}+\boldsymbol{\pi}\\ &= -\frac{\mu}{qd_0}\begin{pmatrix} d_0 & \pi_1\end{pmatrix}P\begin{pmatrix} x & 0\\ 0 & \frac{e^{\gamma x}-1}{\gamma}\end{pmatrix}P^{-1}+\boldsymbol{\pi}.\end{aligned}$$

Since $P^{-1} = \frac{1}{\mu}\begin{pmatrix} \pi_1 & \pi_0\\ -d_0 & d_1\end{pmatrix}$, we get:

$$\begin{aligned}\boldsymbol{F}(\mathrm{x}) &= -\frac{1}{qd_0}\begin{pmatrix} d_0 x & \pi_1\frac{e^{\gamma x}-1}{\gamma}\end{pmatrix}\begin{pmatrix} \pi_1 & \pi_0\\ -d_0 & d_1\end{pmatrix}+\boldsymbol{\pi}\\ &= (1-\frac{x}{q})\boldsymbol{\pi}+\pi_1\frac{e^{\gamma x}-1}{\gamma}\begin{pmatrix} 1 & -\frac{d_1}{d_0}\end{pmatrix}.\end{aligned}$$

The first part of the proof is then achieved, it remains to prove the case $x > q$ for which Identity (4) will be of great interest:

$$\begin{aligned}\boldsymbol{F}(x) &= \boldsymbol{F}(q)\exp\left((x-q)M\right)\\ &= \pi_1\frac{e^{\gamma q}-1}{\gamma q}\begin{pmatrix} 1 & -\frac{d_1}{d_0}\end{pmatrix}\exp\left((x-q)M\right)\\ &= \pi_1\frac{e^{\gamma q}-1}{\gamma q}\begin{pmatrix} 1 & -\frac{d_1}{d_0}\end{pmatrix}P\begin{pmatrix} 1 & 0\\ 0 & e^{\gamma(x-q)}\end{pmatrix}P^{-1}.\end{aligned}$$

Taking into account of the equality $d_0\pi_0 + d_1\pi_1 = \mu$, we obtain:

$$\begin{aligned}\boldsymbol{F}(x) &= \pi_1\frac{e^{\gamma q}-1}{\gamma q}\begin{pmatrix} 0 & -\frac{\mu}{d_0}\end{pmatrix}\begin{pmatrix} 1 & 0\\ 0 & e^{\gamma(x-q)}\end{pmatrix}P^{-1}\\ &= \pi_1\frac{e^{\gamma q}-1}{\gamma q}\begin{pmatrix} 0 & -\frac{\mu}{d_0}e^{\gamma(x-q)}\end{pmatrix}P^{-1}\\ &= \pi_1\frac{e^{\gamma x}-e^{\gamma(x-q)}}{\gamma q}\begin{pmatrix} 1 & -\frac{d_1}{d_0}\end{pmatrix}.\end{aligned}$$

■

The stability condition (1) forces the parameter γ to be negative. The two-state example was studied by Kulkarni (2007). From a mathematical point of view, its solution is naturally equal to the one proposed in Theorem 3.1. The difference lies in the proof and also in the solution expression which is less crowded in Theorem 3.1.

General case

In this subsection, we give an expression of the unique solution of Problem $(\mathcal{P})$. This solution is based on the technique of randomization which is acknowledged by its numerical stability, since it involves only nonnegative real numbers bounded by 1. It is well-known that the matrix $P = \frac{A}{\lambda}+I$ is stochastic, where $\lambda = \max\{-a_{ii} \,/\, i \in S\}$ and I is the identity matrix. Let d be the smallest positive effective input rate: $d = \min\{d_i \,/\, i \in S_+\}$. The next theorem gives an analytic solution of Problem $(\mathcal{P})$ without any spectral analysis on the matrix $M = AD^{-1}$. In contrast with previous works, this result does not distinguish between the case where D is invertible and D is singular.

Theorem 3.2

The expression of the solution of Problem $(\mathcal{P})$ *is:*

For $x \in [0,q)$

$$\boldsymbol{F}(x) = \frac{d}{\lambda q}\sum_{n\ge1} e^{-\frac{\lambda x}{d}}\frac{\left(\frac{\lambda x}{d}\right)^n}{n!}\sum_{k=0}^{n-1}\boldsymbol{b}(k)+(1-\frac{x}{q})\boldsymbol{\pi}$$

For $x \ge q$

$$\boldsymbol{F}(x)=\frac{d}{\lambda q}\sum_{n\ge1}e^{-\frac{\lambda x}{d}}\frac{\left(\frac{\lambda x}{d}\right)^n}{n!}\sum_{k=0}^{n-1}\boldsymbol{b}(k)-\sum_{n\ge1}e^{-\frac{\lambda(x-q)}{d}}\frac{\left(\frac{\lambda(x-q)}{d}\right)^n}{n!}\sum_{k=0}^{n-1}\boldsymbol{b}(k)$$

where $\boldsymbol{b}(k)=\lim_{n\to\infty}\boldsymbol{b}(n,k)$ *and* $\left(\boldsymbol{b}(n,k):=(b_i(n,k), i\in S)\right)_{n\ge k}$ *is the sequence defined by the following recursive expressions:*

- For $i\in S_+$

$b_i(n,0)=\pi_i$ and for $k=1,\dots,n$

$$b_i(n,k)=\left(1-\frac{d}{d_i}\right)b_i(n,k-1)+\frac{d}{d_i}\sum_{j\in S}b_j(n-1,k-1)p_{ji}$$

- For $i\in S_-\cup S_0$

$b_i(n,n)=0$ and for $k=n-1,\dots,0$

$$b_i(n,k)=\frac{-d_i}{d-d_i}b_i(n,k+1)+\frac{d}{d-d_i}\sum_{j\in S}b_j(n-1,k)p_{ji}$$

Proof: According to Theorem 2.1, it is sufficient to prove that $\boldsymbol{F}(x)$ satisfies all the conditions of Problem $(\mathcal{P})$. For the last boundary condition, it is clear that $\sum_{i\in S}F_i(0)=1$, since $\sum_{i\in S}\pi_i=1$.

- $x\in(0,q)$

$$\boldsymbol{F}'(x)=-\frac{1}{q}\sum_{n\ge1}e^{-\frac{\lambda x}{d}}\frac{\left(\lambda x/d\right)^n}{n!}\sum_{k=0}^{n-1}\boldsymbol{b}(k)+\frac{1}{q}\sum_{n\ge0}e^{-\frac{\lambda x}{d}}\frac{\left(\lambda x/d\right)^n}{n!}\sum_{k=0}^{n}\boldsymbol{b}(k)-\frac{\boldsymbol{\pi}}{q}$$

At the point $x=0$, the above equality gives $\boldsymbol{F}'(0)=\frac{\boldsymbol{b}(0)}{q}-\frac{\boldsymbol{\pi}}{q}$. Since $b_i(n,0)=\pi_i$ for all $i\in S_+$, the limit on the index n leads to $b_i(0)=\pi_i$ and therefore $F_i'(0)=0$, for all $i\in S_+$. By taking into account the fact that $\sum_{n\ge0}e^{-\frac{\lambda x}{d}}\frac{\left(\lambda x/d\right)^n}{n!}=1$, the difference between $\boldsymbol{F}'(x)$ and $\boldsymbol{F}'(0)$ fulfills:

$$\begin{aligned}\boldsymbol{F}'(x)-\boldsymbol{F}'(0) &= -\frac{1}{q}\sum_{n\ge1}e^{-\frac{\lambda x}{d}}\frac{\left(\lambda x/d\right)^n}{n!}\sum_{k=0}^{n-1}\boldsymbol{b}(k)\\ &+\frac{1}{q}\sum_{n\ge0}e^{-\frac{\lambda x}{d}}\frac{\left(\lambda x/d\right)^n}{n!}\left[\sum_{k=0}^{n}\boldsymbol{b}(k)-\boldsymbol{b}(0)\right]\\ &=\frac{1}{q}\sum_{n\ge1}e^{-\frac{\lambda x}{d}}\frac{\left(\lambda x/d\right)^n}{n!}\sum_{k=0}^{n-1}\left[\boldsymbol{b}(k+1)-\boldsymbol{b}(k)\right]\end{aligned}\qquad(5)$$

The recursive formula related to the sequence $\left(b_i(n,k)\right)_{n\ge k}$ given separately for $i\in S_+$ and $i\in S_-\cup S_0$ can be lumped into one expression:

$$d_ib_i(n,k)=(d_i-d)b_i(n,k-1)+d\sum_{j\in S}b_j(n-1,k-1)p_{ji}$$

By taking the limit over n:

$$d_i\left(b_i(k)-b_i(k-1)\right)=d\sum_{j\in S}b_j(n-1,k-1)\left[p_{ji}-\delta_{ij}\right],$$

Where δ_{ij} is the Kronecker symbol. In matrix form, the above equality can be written as follows:

$$\left(\boldsymbol{b}(k)-\boldsymbol{b}(k-1)\right)D=d\,\boldsymbol{b}(k-1)\left[P-I\right].\qquad(6)$$

Merging this last equality in Identity (5), we get:

$$\begin{aligned}\left(\boldsymbol{F}'(x)-\boldsymbol{F}'(0)\right)D &= \frac{1}{q}\sum_{n\ge1}e^{-\frac{\lambda x}{d}}\frac{\left(\lambda x/d\right)^n}{n!}\sum_{k=0}^{n-1}\left[\boldsymbol{b}(k+1)-\boldsymbol{b}(k)\right]D\\ &=\frac{d}{q}\sum_{n\ge1}e^{-\frac{\lambda x}{d}}\frac{\left(\lambda x/d\right)^n}{n!}\sum_{k=0}^{n-1}\boldsymbol{b}(k)\left[P-I\right].\end{aligned}$$

Replacing the matrix $P-I$ by its value $\frac{A}{\lambda}$, it comes that:

$$\begin{aligned}\left(\boldsymbol{F}'(x)-\boldsymbol{F}'(0)\right)D &= \frac{d}{\lambda q}\sum_{n\ge1}e^{-\frac{\lambda x}{d}}\frac{\left(\lambda x/d\right)^n}{n!}\sum_{k=0}^{n-1}\boldsymbol{b}(k)A\\ &=\boldsymbol{F}(x)A,\text{ since }\boldsymbol{\pi}A=\boldsymbol{0}.\end{aligned}$$

Now, let us deal with the case $x>q$.

- $x\in(q,+\infty)$

$$\begin{aligned}\boldsymbol{F}'(x) &= -\frac{1}{q}\sum_{n\ge1}e^{-\frac{\lambda x}{d}}\frac{\left(\lambda x/d\right)^n}{n!}\sum_{k=0}^{n-1}\boldsymbol{b}(k)\\ &+\frac{1}{q}\sum_{n\ge0}e^{-\frac{\lambda x}{d}}\frac{\left(\lambda x/d\right)^n}{n!}\sum_{k=0}^{n}\boldsymbol{b}(k)\\ &+\frac{1}{q}\sum_{n\ge1}e^{-\frac{\lambda(x-q)}{d}}\frac{\left(\lambda(x-q)/d\right)^n}{n!}\sum_{k=0}^{n-1}\boldsymbol{b}(k)\\ &-\frac{1}{q}\sum_{n\ge0}e^{-\frac{\lambda(x-q)}{d}}\frac{\left(\lambda(x-q)/d\right)^n}{n!}\sum_{k=0}^{n}\boldsymbol{b}(k)\\ &=\frac{1}{q}\sum_{n\ge0}e^{-\frac{\lambda x}{d}}\frac{\left(\lambda x/d\right)^n}{n!}\boldsymbol{b}(n)\\ &-\frac{1}{q}\sum_{n\ge0}e^{-\frac{\lambda(x-q)}{d}}\frac{\left(\lambda(x-q)/d\right)^n}{n!}\boldsymbol{b}(n)\end{aligned}$$

By adding and subtracting the same term $\boldsymbol{b}(0)$, the previous equality becomes:

$$\boldsymbol{F}'(x) = \frac{1}{q}\sum_{n\geq 0} e^{-\frac{\lambda x}{d}} \frac{\left(\lambda x/d\right)^n}{n!} \sum_{k=0}^{n-1}[\boldsymbol{b}(k+1) - \boldsymbol{b}(k)]$$

$$-\frac{1}{q}\sum_{n\geq 0} e^{-\frac{\lambda(x-q)}{d}} \frac{\left(\lambda(x-q)/d\right)^n}{n!} \sum_{k=0}^{n-1}[\boldsymbol{b}(k+1) - \boldsymbol{b}(k)].$$

Taking into account of Identity (6), the same mathematical development for the case $x \in (0,q)$ leads to the required result $\boldsymbol{F}'(x)D = \boldsymbol{F}'(x)\mathrm{A}$. To achieve the proof of this theorem, it remains to prove the condition $\lim_{x\to\infty} F_i(x) = 0, \forall i \in S$. This property holds giving that the series $\sum_{k\geq 0}\boldsymbol{b}(k)$ converges (see the appendix for the proof) and from the fact that:

$\lim_{x\to\infty}\sum_{n\geq 0} e^{-\frac{\lambda x}{d}} \frac{\left(\lambda x/d\right)^n}{n!} a_n = \lim_{n\to\infty} a_n$, for all convergent sequence $(a_n)_{n\geq 0}$.

This last result is classical and its proof is easy.

■

The matrix P is stochastic, all its entries are nonnegative and the sum of each row is equal to 1. According to the recursive formula, each element $b_i(n,k)$ is a convex combination of two elements in [0,1]. It is easy to prove by induction that the sequence $(b_i(n,k))_{n\geq k}$ is a nonnegative sequence bounded by π_i. Also, the terms $e^{-\frac{\lambda x}{d}} \frac{\left(\lambda x/d\right)^n}{n!}$ are actually the Poisson distribution. All these remarks ensure the numerical stability of the proposed method since it involves only positive elements bounded by 1. On the other hand, the sequence $(b_i(n,k))_{n\geq k}$ is exactly the same one used in Nabli (2004) for general stochastic fluid models.

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Appendix

The objective of this appendix is to proof the convergence of the series $\sum_{k\geq 0}\boldsymbol{b}(k)$. Considering that $P - I = \frac{A}{\lambda}$, Identity (6) is equivalent to:

$$\boldsymbol{b}(k)D = \boldsymbol{b}(k-1)\left(D + \frac{d}{\lambda}A\right)$$

$$\Leftrightarrow \boldsymbol{b}(k) = \boldsymbol{b}(k-1)\left(I + \frac{d}{\lambda}AD^{-1}\right)$$

$$\Leftrightarrow \boldsymbol{b}(k) = \boldsymbol{b}(0)\mathrm{C}^k,$$

where $C = I + \frac{d}{\lambda}AD^{-1}$. Nabli (2004) has proved that the sequence $(\boldsymbol{b}(k))_{k\geq 0}$ converges to $\mathbf{0}$. Thus $\boldsymbol{b}(0)$ is orthogonal to all eigenvectors of C associated to eigenvalues θ satisfying $|\theta| \geq 1$. So, the series $\sum_{k\geq 0}\boldsymbol{b}(k)$ converges since only eigenvalues of module less than 1 are involved. Here it is supposed implicitly that D is invertible $(S_0 = \varnothing)$. For the case where D is singular, the partition $S = \hat{S} \cup S_0$ must be considered. The proof follows the same demarche made in Theorem 2.2.

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حل مقارب للنماذج العشوائية السائلة ذات القفزات التصاعدية

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الملخص

أنموذج العملية العشوائية ذات القفزات التصاعدية يُستعمل غالبًا في نظم الإنتاج والمخزون المعدلة ببيئة ماركوفية. عملية السوائل تمثل مستوى المخزون والبيئة الماركوفية توصف مراحل الإنتاج والمبيعات. التوزيع الثنائي لمستوى السوائل والبيئة الماركوفية يحقق نظامين لمعادلات تفاضلية جزئية مع شروط حدية خاصة. في هذه الورقة البحثية تم إثبات وحدانية الحل ومن ثم اقتراح حلًّا تتوفر فيه شروط الاستقرار العددي والدقة.

الكلمات المفتاحية: عملية ماركوف، المعادلات التفاضلية الجزئية، نماذج السوائل العشوائية.