Nonconvex Sweeping Process with Non-Compact Valued Perturbation and with Delay

Ahmed Gamal Ibrahim and Feryal A. Aladsani

Department of Mathematics, Faculty of Science King Faisal University
Al-Hassa, Saudi Arabia.

Received 25 October 2015 - Accepted 29 May 2016

ABSTRACT

The sweeping processes correspond to several important mechanical problems. Therefore, the existence theorems of solutions for sweeping processes have been attracted the attention by many authors.

In this paper, we aim to prove existence theorems concerning the existence of solutions for functional differential inclusions governed by sweeping process with non- compact valued perturbations.

Depending on a discretization technique used in recent papers, we prove two new existence results of solutions for first and second order functional differential inclusions governed by sweeping process with non- compact valued perturbations, where the moving set is a multifunction depending on time and state and with nonempty closed uniformly ρ-prox-regular values. We do not assume that the values of the moving set are contained in a fixed compact subset. In addition, our technique allows us to discuss some sweeping process problems with noncompact perturbations.

Key Words: Differential inclusion, Normal cone, Sweeping processes.

INTRODUCTION

Moreau, 1979, initiated the existence of solutions for sweeping processes. Since then, important improvements have been developed by weaken assumptions in order to obtain the most general result of existence for sweeping processes, see (Thibault, 2003, Castaing, *et al.* 2009, Aitalioubrahim, 2012, Gomaa, 2013, Haddad and Haddad, 2014).

Let $r>0$, $C_r = C([-r,0],H)$ be the Banach space of continuous functions from $[-r,0]$ to H endowed with the uniform norm, $\Gamma, K:[-r,0]\times H \to 2^H$ multifunctions with nonempty, closed and uniformly ρ-prox-regular values $(\rho>0)$, $F: I\times C_r \to 2^H$ a multifunction with closed and convex(not necessarily compact)values. For each $t\in I$, define $\tau(t): C([-r,t],H)\to C_r$ by $(\tau(t))g(s) = g(t+s), \forall s\in[-r,0]$. Let $\Psi,\Phi\in C_r$ be fixed two functions. Consider the following first and second order nonconvex sweeping processes with delay:

$$\begin{cases} u(t)=\Psi(t),\ t\in[-r,0]; \\ u(t)=\Psi(0)+\int_0^t u'(s)ds,\ t\in I; \\ u(t)\in\Gamma(t,u(t)),\ t\in I: \\ -u'(t)\in N_{\Gamma(t,u(t))}(u(t))+F(t,\tau(t)u),\ a.e.\ t\in I, \end{cases} \quad (1)$$

and

$$\begin{cases} u(t)=\Psi(t),\ \ t\in[-r,0]; \\ u(t)=\Psi(0)+\int_0^t u'(s)ds,\ t\in I, \\ p(t)=\Phi(0)+\int_0^t u(s)ds,\ t\in I, \\ \quad t,p \\ u(t)\in K(t,u(t)),\ t\in I; \\ u'(t)\in N_{K(t,p(t))}(u(t))+F(t,\tau(t)u), a.e. t\in I \end{cases} \quad (2)$$

Several authors under different assumptions have studied the existence of solutions for the problems(1) and (2). For example: Chemetov and Monteiro Marques, 2007, considered the problem(1) without delay and when Γ taking a closed uniformly ρ-proxy regular (ρ>0) values and F is Crartheodary with compact convex values. Castaing (2009), established the existence of solutions for the problem (2) without delay and in the case when K is a Lipschitz multifunction with closed uniformly ρ-proxy regular (ρ>0) values, and $F: I\times H\times H\to 2^H$ is upper semi continuous with nonempty convex compact values. Aitalioubrahim, 2012, considered the problem(1) when $\Gamma: I\to 2^H$ is a multifunction (Γ depends on the time only) and taking nonempty compact, uniformly ρ-proxy regular (ρ>0) values in H. Haddad (2013), established the existence of solutions for the problem(1) without delay and when Γ is a Lipschitz multifunction with closed uniformly ρ-proxy regular (ρ>0) values such that $\Gamma(t,x) \subseteq Z$, for all $(t,x) \in I \times H$ for some fixed compact set Z, and F is a convex weakly compact valued multifunction. Haddad and Haddad (2014), considered the problem(1) when the values of F are convex and weakly compact and for any convergent sequence (t_n) in I and for any bounded set $A \subseteq H$, the set $\bigcup\{\Gamma(t_n,x): n \geq 1, x \in A\}$ is ball compact. Noel and Thibault (2014), established the existence of solutions of the problem(1) without delay and the values of Γ are closed uniformly ρ-proxy regular (ρ>0) and ball compact. For other contributions on differential inclusions, see (Gomaa, 2013).

Motivated by these works, in this paper, we prove the existence of solutions of (1.2) and (1.3) in the case when $\Gamma, K : [-r,0] \times H \to 2^H$ are multifunctions and taking a closed ρ-proxy regular (ρ>0) values in H (not necessarily ball compact), and $F : I \times C_r \to 2^H$ is a multifunction with nonempty closed (not necessarily compact) values. We assume that neither the values of Γ nor the values of K are contained in a fixed compact subset. Instead, we suppose that both Γ and K satisfying a condition contains a measure of noncompactness. In addition, our technique allows us to discuss some sweeping process problems with non-compact valued perturbations.

MATERIALS AND METHODS

In order to achieve our goals we use the discretization technique used in Castaing *et al*. (2009), and Aitalioubrahim (2012), with the following known definition and facts:

Definition 2.1. (Edmond, and Thibault, 2006) "If S is a closed subset of H and $x \in H$, then the proximal normal cone of S at x is defined by:

$$N_S^p(x) = \{y \in H : \exists \lambda > 0 \textit{ such that } x \in P_S(x + \lambda y)\}$$

where $P_S(z)$ is the projection of the point z on S."

It is known (see for example, Castaing *et al*. 2009, Haddad, 2013,) that "$N_S^p(x) = \partial^p \psi_S(x)$, where $\partial^p \psi_S(x)$ is the proximal subdifferential of the indicator function $\psi_S(x)$ of S, i.e. $\psi_S(x) = 0$ if $x \in S$ and ∞ otherwise".

Definition 2.2 (Edmond, and Thibault, 2006) "For a given $\rho \in]0,\infty]$, a subset S is said to be uniformly ρ-prox-regular, if and only if every nonzero proximal normal vector can be realized by a ρ-ball".

For more information about uniformly ρ-prox-regular sets, we refer the reader to (Edmond, and Thibault, 2006, Castaing *et al*. 2009, and Haddad, 2013).

Lemma 1. (Edmond, and Thibault, 2006, Castaing *et al*. 2009, Haddad, 2013). "Let S be a nonempty closed subset in H. The following assertions are satisfied.

1. For any $x \in S$, $N_S^p(x) \cap B = \partial^p d_S(x)$, where the function $d_S(x)$ is the distance function and $B = \{x \in H : \|x\| \leq 1\}$.
2. If S is uniformly ρ-prox-regular subset in H where $\rho \in]0,\infty]$, and $x \in H$ such that $d_S(x) < \rho$, then

   (i) $P_S(x)$ is a singleton.

   (ii) $\partial^p d_S(x) = \partial^c d_S(x)$, where, $\partial^c d_S(x)$ is the Clark subdifferential. So, in such case the subdifferential is a closed convex subset in H."

As a consequence of (ii) we get that for nonempty closed uniformly ρ-prox-regular sets S and $x \in H$ such that $d_S(x) < \rho$, then $N_S^p(x) = N_S(x)$.

Lemma 2. (Prop. 2.2. Haddad, 2013). "Let $\rho \in]0,\infty]$, Ω be an open subset in H, and C be a Hausdorff continuous set-valued mapping defined on Ω and with nonempty closed uniformly ρ-prox-regular values in H . Then, for a given η∈ (0, ρ) the following holds: for any $z \in \Omega$, $x \in C(z)+(\rho-\eta)B$,$x_n \to x$ $z_n \to z$ with $z_n \in \Omega$ and $y_n \in \partial^p_{C(z_n)}(x_n)$ such that (y_n) converges weakly to y one has $y \in \partial^p_{C(z)}(x)$."

Lemma 3. (lemma 3.2, Zhu, 1991). "Let X be a separable Banach space, $G:[a,b]\to 2^X$ a measurable multifunction with nonempty closed values and $z:[a,b]\to X$ a measurable function. Then, for any measurable function $r:[a,b]\to[0,\infty)$ there exists a measurable selection g of G such that

$\| g(t)-z(t)\| = d(z(t),G(t))+r(t),\ a.e.\ t\in[a,b].$

RESULTS

In this section we prove two existence results of solutions for the problems (1) and (2).

1-Existence result of solutions for the problem(1).

Theorem 1. Let H be a real separable Hilbert space, $\Gamma: I\times H\to 2^H$ be a multifunction with nonempty closed uniformly ρ-prox-regular values and $F: I\times C_r\to 2^H$ be a multifunction with nonempty closed values. We assume the following conditions:

(H1) There is a positive real number k such that: $\Gamma(t,x)\subseteq kB,\ \forall (t,x)\in I\times H$.

(H2) There are two constants $L_1>0$ and $L_2\in(0,1)$ such that for all $t,s\in I$ and $x,u,v\in H$

$| d_{\Gamma(t,u)}(x)-d_{\Gamma(s,v)}(x)| \le L_1|t-s|+L_2|u-v|$.

(H3) For any $t\in I$ and any bounded subset A in H with $\chi(A)>0$ and any $r>0$ one has, $\chi(\Gamma(t,A)\cap rB)<\chi(A)$,

where, χ is the Hausdorff measure of noncompactness on H.

(H4) For any $\psi\in C_r$, the function $t\to F(t,\psi)$ is measurable and there is $m\in L^1(I,R^{\ge 0})$ such that for any $t\in I$ and $\psi_1,\psi_2\in C_r$,

$h(F(t,\psi_1),F(t,\psi_2))\le m(t)\|\psi_1-\psi_2\|_{C_r}$.

(H5) There exists a continuous function $w: I\to R^{\ge 0}$ such that for all $\psi\in C_r$, and all $t\in I$, $F(t,\psi)\subseteq w(t)(1+\|\psi(0)\|)B.$

Then, for any fixed $\Psi\in C_r$, with $\Psi(0)\in\Gamma(0,\Psi(0))$, there is a continuous function $u:[-r,T]\to H$ such that u is Lipschitz on $[0,T]$ and satisfies (1).

Proof. Let $n_0\ge 2$ be a natural number satisfying

$$\frac{L_1+2L_3(1+k)}{1-L_2}\,\frac{T}{2^{n_0}}\le\frac{\rho}{2}, \tag{4}$$

where, $L_3=\max_{t\in I} w(t)$. For any natural number $n\ge n_0$ and any $i=0,1,2,\ldots,2^n-1$, we consider the points, $t_i^n = i\,\mu_n$, $\mu_n=\frac{T}{2^n}$. Also, let $\theta_n,\delta_n: I\to I$ be such that: $\theta_n(0)=0$, $\theta_n(t)=t^n_{i+1}$, $t\in(t_i^n,t^n_{i+1}]$, $\delta_n(t)=t_i^n$, $t\in[t_i^n,t^n_{i+1})$, $\delta_n(T)=T$, $i=0,1,2,\ldots,2^n-1$.

Step 1. We show that if $n\ge n_0$ is a fixed natural number and $h\in L^1(I,H)$, then there are $u_n\in C([-r,T],H)$ and $g_n\in L^1(I,H)$ such that u_n is absolutely continuous on I and the following properties hold:

$$\begin{cases} (i)\ u_n(t)=\psi(t),\ t\in[-r,0];\\ (ii)\ u_n(\theta_n(t))\in\Gamma(\theta_n(t),u_n(\delta_n(t))),\ t\in I;\\ (iii)\ g_n(t)\in F(t,\tau(\delta_n(t))u_n),\ t\in I;\\ (iv)\ \|g_n(t)-h(t)\|\le d(h(t),F(t,\tau(\delta_n(t))u_n))+\frac{1}{n^2},\ a.e.\ t\in I;\\ (v)\ -u_n'(t)-g_n(t)\in N_{\Gamma(\theta_n(t),u_n(\delta_n(t)))}(u_n(\theta_n(t)))\cap\gamma_1 B,\ a.e.\ t\in I, \end{cases} \tag{5}$$

where, $\gamma_1=\frac{L_1+2L_3(1+k)}{1-L_2}$. In view of Lemma 3, there is a function $g_0^n\in L^1([t_0^n,t_1^n],H)$ such that $g_0^n(t)\in F(t,\tau(0)\psi),\ a.e.\ t\in[t_0^n,t_1^n]$ and

$$\|g_0^n(t)-h(t)\|\le d(h(t),F(t,\tau(0)\psi))+\frac{1}{n^2},\ a.e.\ t\in[t_0^n,t_1^n]. \tag{6}$$

We put $x_0^n = \psi(0) \in \Gamma(0, x_0^n)$ and (7)

$$x_1^n = proj_{\Gamma(t_1^n, x_0^n)}(x_0^n - \int_{t_0^n}^{t_1^n} g_0^n(s)ds).$$

Let us show that x_1^n is well defined. By (H2) and (H5) we have

$$d_{\Gamma(t_1^n,x_0^n)}(x_0^n - \int_{t_0^n}^{t_1^n} g_0^n(s)ds) \le d_{\Gamma(t_1^n,x_0^n)}(x_0^n) + \int_{t_0^n}^{t_1^n} \| g_0^n(s)\| ds$$
$$\le | d_{\Gamma(t_1^n,x_0^n)}(x_0^n) - d_{\Gamma(t_0^n,x_0^n)}(x_0^n)| + \int_{t_0^n}^{t_1^n} L_3(1+\|(\tau(0)\psi)(0)\|)ds$$
$$\le L_1 |t_1^n - t_0^n| + \mu_n L_3(1+\|\psi(0)\|)$$
$$\le [L_1 + L_3(1+k)]\frac{T}{2^n} \le (\frac{L_1+L_3(1+k)}{1-L_2})\frac{T}{2^{n_0}} \le \frac{\rho}{2} < \rho. \quad (8)$$

As $\Gamma(t_0^n, x_0^n)$ is uniformly ρ-prox-regularity, by Lemma 1 and (8), x_1^n is well defined.

Then, we can define for $t \in [t_0^n, x_0^n]$,

$$u_n(t) = x_0^n + \frac{t - t_0^n}{\mu_n}[x_1^n - (x_0^n - \int_{t_0^n}^{t_1^n} g_0^n(s)ds)] - \int_{t_0^n}^{t} g_0^n(s)ds. \quad (9)$$

Observe that this relation yields

$$u_n'(t) = x_1^n - (x_0^n - \int_{t_0^n}^{t_1^n} g_0^n(s)ds) - g_0^n(t), \text{ a. e. } for\ t \in [t_0^n, t_1^n]$$

So, by (7)

$$u_n'(t) + g_0^n(t) \in -N^p_{\Gamma(\theta_n(t) u_n(\delta_n(t)))}(u_n(\theta_n(t)), \text{ a. e. } for\ t \in [t_0^n, t_1^n]. \quad (10)$$

Moreover, the condition (H2) tells us

$$\| x_1^n - x_0^n \| \le \| x_1^n - (x_0^n - \int_{t_0^n}^{t_1^n} g_0^n(s)ds)\| + \int_{t_0^n}^{t_1^n} \| g_0^n(s)\| ds$$
$$= d_{\Gamma(t_1^n,x_0^n)}(x_0^n - \int_{t_0^n}^{t_1^n} g_0^n(s)ds) + \int_{t_0^n}^{t_1^n} \| g_0^n(s)\| ds$$
$$\le d_{\Gamma(t_1^n,x_0^n)}(x_0^n) + 2\int_{t_0^n}^{t_1^n} \| g_0^n(s)\| ds$$
$$= | d_{\Gamma(t_1^n,x_0^n)}(x_0^n) - d_{\Gamma(t_0^n,x_0^n)}(x_0^n)| + 2\int_{t_0^n}^{t_1^n} \| g_0^n(s)\| ds$$
$$\le L_1 \mu_n + 2\mu_n L_3(1+k).$$

Next, by induction, we can define for $i = 0,1,2,...,2^n - 1$ and for $t \in [t_i^n, t_{i+1}^n]$,

$$u_n(t) = x_i^n + \frac{t - t_i^n}{\mu_n}[x_{i+1}^n - (x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds)] - \int_{t_i^n}^{t} g_i^n(s)ds, \quad (12)$$

where

$$x_{i+1}^n = proj_{\Gamma(t_{i+1}^n, x_i^n)}(x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds)), \quad (13)$$

$$g_i^n \in L^1([t_i^n, t_{i+1}^n], H),\ g_i^n(t) \in F(t, \tau(\delta_n(t))u_n),\ t \in [t_i^n, t_{i+1}^n], \quad (14)$$

and

$$\| g_i^n(t) - h(t)\| \le d(h(t), F(t, \tau(\delta_n(t))u_n)) + \frac{1}{n^2},\ a.\ e.\ t \in [t_i^n, t_{i+1}^n]. \quad (15)$$

Let us show that x_{i+1}^n, given by (13), is well define. By construction one obtains

$$d_{\Gamma(t_{i+1}^n,x_i^n)}(x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds)$$
$$\le d_{\Gamma(t_{i+1}^n,x_i^n)}(x_i^n) + \int_{t_i^n}^{t_{i+1}^n} \| g_i^n(s)\| ds$$
$$\le | d_{\Gamma(t_{i+1}^n,x_i^n)}(x_i^n) - d_{\Gamma(t_i^n,x_{i-1}^n)}(x_i^n)| + \int_{t_i^n}^{t_{i+1}^n} \| g_i^n(s)\| ds$$
$$\le | d_{\Gamma(t_{i+1}^n,x_i^n)}(x_i^n) - d_{\Gamma(t_i^n,x_{i-1}^n)}(x_i^n)| + \int_{t_i^n}^{t_{i+1}^n} L_3(1+\|\tau(\delta_n(t))u_n)(0)\| ds$$
$$\le L_1 |t_{i+1}^n - t_i^n| + L_2 \| x_i^n - x_{i-1}^n \| + \mu_n L_3(1+\|u_n(\delta_n(t))\|)$$
$$\le \mu_n[L_1 + L_3(1+k)] + L_2 \| x_i^n - x_{i-1}^n \|. \quad (16)$$

Also, we have

$$\| x_i^n - x_{i-1}^n \| \le \| x_i^n - (x_{i-1}^n - \mu_n g_{i-1}^n(t_{i-1}^n))\| + \mu_n \| g_{i-1}^n(t_{i-1}^n)\|$$
$$= d_{\Gamma(t_i^n,x_{i-1}^n)}(x_{i-1}^n - \mu_n g_{i-1}^n(t_{i-1}^n)) + \mu_n L_3(1+k)$$
$$\le d_{\Gamma(t_i^n,x_{i-1}^n)}(x_{i-1}^n) + \mu_2 \| g_{i-1}^n(t_{i-1}^n))\| + \mu_n L_3(1+k)$$
$$\le | d_{\Gamma(t_i^n,x_{i-1}^n)}(x_{i-1}^n) - d_{\Gamma(t_{i-1}^n,x_{i-2}^n)}(x_{i-1}^n)| + 2\mu_2 L_3(1+k)$$
$$\le \mu_2(L_1 + 2L_3(1+k)) + L_2 \| x_{i-1}^n - x_{i-2}^n \|. \quad (17)$$

By induction, from (11) and (17) we get

$$\| x_i^n - x_{i-1}^n \|$$
$$\le \mu_n(L_1 + 2L_3(1+k)) + L_2[\mu_n(L_1 + 2L_3(1+k)) + L_2 \| x_{i-2}^n - x_{i-3}^n \|$$
$$= \mu_n(L_1 + 2L_3(1+k))[1 + L_2] + L_2^2 \| x_{i-2}^n - x_{i-3}^n \|$$
$$\le \mu_n(L_1 + 2L_3(1+k))[1 + L_2 + L_2^2 + ... + L_2^{i-1}] + L_2^i \| x_1^n - x_0^n \|$$
$$\le \mu_n(L_1 + 2L_3(1+k))[1 + L_2 + L_2^2 + ... + L_2^{i-1} + L_2^i]. \quad (18)$$

Since $L_2 \in (0,1)$, then (16) and (18) yield,

$$d_{\Gamma(t_{i+1}^n,x_i^n)}(x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds)$$
$$\le \mu_n(L_1 + 2L_3(1+k))[1 + [1 + L_2 + L_2^2 + ... + L_2^{i-1} + L_2^i + L_2^{i+1}]$$
$$\le \mu_n(L_1 + 2L_3(1+k))\frac{1}{1-L_2}. \quad (19)$$

Therefore,

$$d_{\Gamma(t_{i+1}^n,x_i^n)}(x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds) \le (L_1 + 2L_3(1+k))\frac{T}{(1-L_2)2^n}$$
$$\le \frac{L_1 + 2L_3(1+k)}{1-L_2}\frac{T}{2^{n_0}} \le \frac{\rho}{2} < \rho.$$

As Γ has uniformly ρ-prox-regular values, by Lemma 1 and (19), x_{i+1}^n is well defined. Next, let $i \in \{0,1,2,...,2^n - 1\}$ be fixed. By (12), for almost $t \in [t_i^n, t_{i+1}^n]$ one obtains

$$u_n'(t) = \frac{1}{\mu_n}[x_{i+1}^n - (x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds)] - g_i^n(t). \quad (20)$$

From this relation and (19) one obtains

$$\| u_n'(t) + g_i^n(t)\| \le \frac{1}{\mu_n}\| x_{i+1}^n - (x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds)\|$$
$$\le \frac{1}{\mu_n} d_{\Gamma(t_{i+1}^n,x_i^n)}(x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds) \quad (21)$$
$$\le (L_1 + 2L_3(1+k))\frac{1}{1-L_2} = \gamma_1.$$

Now, let $g_n : I \to H$ be such that $g_n(t) = g_i^n(t)$, $t \in [t_i^n, t_{i+1}^n)$, $i = 0,1,2,...,2^n - 1$,

$g_n(T) = g^n_{2^n}(T)$. Observe that(13) and (20) imply

$$-u'_n(t) - g_n(t) \in N^P_{\Gamma(\theta_n(t)u_n(\delta_n(t)))}(u_n(\theta_n(t)), \text{a e } for\ t \in I.$$

Step 2. In this step, we show that there are two sequences $(g_n)_{n \geq n_0} \subseteq L^1(I,H)$ and $(u_n)_{n \geq n_0} \subseteq C([-r,H],H)$ such that u_n is absolutely continuous on I and

$$\begin{cases} (i) u_n(t) = \psi(t),\ t \in [-r,0]; \\ (ii) u_n(\theta_n(t)) \in \Gamma(\theta_n(t), u_n(\delta_n(t))),\ t \in I; \\ (iii) g_n(t) \in F(t, \tau(\delta_n(t)) u_n),\ \ t \in I; \\ (iv) \| g_{n+1}(t) - g_n(t) \| \leq d(g_n(t), F(t, \tau(\delta_{n+1}(t)) u_{n+1})) + \frac{1}{(n+1)^2},\ a.e.\ t \in I; \\ (v) - u'_n(t) - g_n(t) \in N^P_{\Gamma(\theta_n(t) u_n(\delta_n(t)))}(u_n(\theta_n(t))) \cap \gamma_1 B(0,1),\ a.e.\ t \in I. \end{cases} \tag{22}$$

Indeed, let h be any fixed element in $L^1(I,H)$, in view of step 1, there are $g_{n_0} \in L^1(I,H)$ and $u_{n_0} \in C([-r,T],H)$ such that u_{n_0} is absolutely continuous on I and the properties (i)-(v) in (5) are satisfied for $n = n_0$. Now, since $g_{n_0} \in L^1(I,H)$, then, in view of step 1, there are $g_{n_0+1} \in L^1(I,H)$ and $u_{n_0} \in C([-r,T],H)$ such that u_{n_0+1} is an absolutely continuous on I and that

$$\begin{cases} (i) u_{n_0+1}(t) = \psi(t),\ t \in [-r,0]; \\ (ii) u_{n_0+1}(\theta_{n_0+1}(t)) \in \Gamma(\theta_{n_0+1}(t), u_{n_0+1}(\delta_{n_0+1}(t))),\ t \in I; \\ (iii) g_{n_0+1}(t) \in F(t, \tau(\delta_{n_0+1}(t)) u_{n_0+1}),\ \ t \in I; \\ (iv) \| g_{n_0+1}(t) - g_{n_0}(t) \| \leq d(g_{n_0}(t), F(t, \tau(\delta_{n_0+1}(t)) u_{n_0+1})) + \frac{1}{(n_0+1)^2},\ a.e.\ t \in I; \\ (v) - u'_{n_0+1}(t) - g_{n_0+1}(t) \in N^P_{\Gamma(\theta_{n_0+1}(t) u_{n_0+1}(\delta_{n_0+1}(t)))}(u_{n_0+1}(\theta_{n_0+1}(t))) \cap \gamma_1 B,\ a.e.\ t \in I. \end{cases} \tag{23}$$

So, we can define inductively two sequences $(g_n)_{n \geq n_0}$ and $(u_n)_{n \geq n_0}$ such that the properties (i)-(v) in (22) are satisfied.

Step 3. Our goal in this step is to prove that the sequence $(u_n|_I)_{n \geq n_0}$ has a subsequence, still denoted by $(u_n|_I)_{n \geq n_0}$, converging uniformly to a Lipschitz function $u \in C(I,H)$. Firstly, we claim that for any $n \geq n_0$, u_n is absolutely on I and

$$\| u'_n(t) \| \leq \frac{L_1 + 2L_3(1+k)}{1 - L_2} + L_3(1+k) := \gamma_2,\ \text{a. e. } t \in I. \tag{24}$$

Indeed, for any $i = 0,1,2,...,2^n - 1$ and any $t_1, t_2 \in (t^n_i, t^n_{i+1})$, $t_1 < t_2$, one obtains from (12), (13) and (19)

$$\begin{aligned} \| u_n(t_1) - u_n(t_2) \| &= \frac{t_2 - t_1}{\mu_n} \| x^n_{i+1} - (x^n_i - \int_{t^n_i}^{t^n_{i+1}} g^n_i(s) ds) \| + \int_{t_1}^{t_2} g^n_i(s) ds) \\ &\leq (\frac{t_2 - t_1}{\mu_n}) d_{\Gamma(t^n_{i+1}, x^n_i)}(x^n_i - \int_{t^n_i}^{t^n_{i+1}} g^n_i(s) ds) + (t_2 - t_1) L_3(1+k) \\ &\leq (t_2 - t_1)[((L_1 + 2L_3(1+k)) \frac{1}{1 - L_2} + L_3(1+k)]. \end{aligned} \tag{25}$$

This means that u_n is absolutely on I and (24) is true. Hence the set $\{u_n|_I : n \geq n_0\}$ is equicontinuous. Let $t \in I$ be a fixed point and $A(t) = \{u_n(t) : n \geq n_0\}$. According to (22) (ii), $u_n(\theta_n(t)) \in \Gamma(\theta_n(t), u_n(\delta_n(t))),\ n \geq n_0$. Then

$$\begin{aligned} u_n(t) &\in \Gamma(\theta_n(t), u_n(\delta_n(t))) + \| u_n(\theta_n(t)) - u_n(t) \| \\ &\subseteq \Gamma(\theta_n(t), u_n(\delta_n(t))) + \gamma_2 | \theta_n(t) - t | \\ &= \Gamma(\theta_n(t), u_n(\delta_n(t))) + \gamma_2 \mu_n. \end{aligned} \tag{26}$$

On the other hand, in view of (H2) for any $n \geq n_0$ and any $z \in \Gamma(\theta_n(t)\ u_n(\delta_n(t)))$ we get

$$\begin{aligned} d_{\Gamma(t, u_n(t))}(z) &= | d_{\Gamma(\theta_n(t), u_n(\delta_n(t)))}(z) - d_{\Gamma(t, u_n(t))}(z) | \\ &\leq L_1 | \theta_n(t) - t | + L_2 \| u_n(\delta_n(t)) - u_n(t) \| \leq L_1 \mu_n + L_2 \gamma_2 \mu_n. \end{aligned}$$

This yields, for $n \geq n_0$, $\Gamma(\theta_n(t)\ u_n(\delta_n(t))) \subseteq \Gamma(t, u_n(t)) + (L_1 \mu_n + L_2 \gamma_2 \mu_n) B$, $\forall t \in I$.

This relation and (26) give us, for $n \geq n_0$

$$u_n(t) \in \Gamma(t, u_n(t)) + \mu_n(\gamma_2 + L_1 + L_2 \gamma_2) B = \Gamma(t, u_n(t)) + \overline{\mu_n} B, \tag{27}$$

where $\overline{\mu_n} = \mu_n(\gamma_2 + L_1 + L_2 \gamma_2)$. Now, assume by contradiction that there is $t_0 \in I$ such that $A(t_0) = \{u_n(t_0) : n \geq n_0\}$ is not relatively compact. Then $\chi(A(t_0)\ > 0$. Observe that, for $n \geq n_0$, $\| u_n(t_0) \| \leq k + \gamma_2 T := \zeta$. So, by (27)

$$u_n(t_0) \in (\Gamma(t_0, u_n(t_0)) \cap \zeta B) + \overline{\mu_n} B,\ \forall n \geq n_0. \tag{28}$$

Moreover, according to (H3), $\chi(A(t_0)\ - \chi(\Gamma(t_0, A(t_0)\ \cap \zeta B) > 0$. Then, we can find $\xi > 0$ such that

$$\chi(A(t_0)) - \chi(\Gamma(t_0, A(t_0)) \cap \zeta B) > 2\xi. \tag{29}$$

Let N_0 be such that $n_0 \leq N_0$ and that $\overline{\mu_n} < \frac{\xi}{2}$

$\forall n \geq N_0$. So by using (28) and (29) one obtains

$$\chi(A(t_0)) = \chi(\{u_n(t_0) : n \geq N_0\})$$
$$\leq \chi(\Gamma(t_0, \{u_n(t_0) : n \geq N_0\}) \bigcap \zeta B) + \chi(\bigcup \{\overline{\mu_n} B : n \geq N_0\})$$
$$\leq \chi(\Gamma(t_0, A(t_0)) \bigcap \zeta B) + \xi < \chi(A(t_0)) - 2\xi + \xi = \chi(A(t_0)) - \xi,$$

which is a contradiction. Therefore, for any $t \in I$, the set $\{u_n(t) : n \geq n_0\}$ is relatively compact. By applying Th. 4, Ch. 1 in (Aubin, and Cellina,1984), there is a Lipschitz function $u : I \to H$, such that the sequence $(u_n|_I)_{n \geq n_0}$ has a subsequence, still denoted by $(u_n|_I)_{n \geq n_0}$, converging uniformly to u. We extend the definition of u on $[-r, T]$ by putting $u(t) = \psi(t)$, $t \in [-r, 0]$. Thus $(u_n)_{n \geq 2}$ converging uniformly to u on $[-r, T]$. In addition, by (3.18), the sequence $(u'_n)_{n \geq n_0}$ is uniformly bounded in $L^2(I, H)$. Hence, without loss of generality, we may suppose that there is $z \in L^2(I, H)$ such that $u'_n \to z$ weakly in $L^2(I, H)$. Then, for each $t \in I$,

$$u(t) = \lim_{n \to \infty} u_n(t) = \Psi(0) + \lim_{n \to \infty} \int_0^t u'_n(s) ds = .$$
$$\Psi(0) + \int_0^{t} z(s) ds$$

u'

Thus $= z$, a.e. So, u'_n converges weakly to u'.

Step 4. In this step we prove that : $u(t) \in \Gamma(t, u(t)), \forall t \in I.$ Let $t \in I$. By (H2) we have

$$d_{\Gamma(t,u(t))}(u(t)) \leq d_{\Gamma(t,u(t))}(u_n(\theta_n(t)) + \|u_n(\theta_n(t)) - u_n(t)\|$$
$$\leq |d_{\Gamma(t,u(t))}(u_n(\theta_n(t)) - d_{\Gamma(\theta_n(t), u_n(\delta_n(t)))}(u_n(\theta_n(t))| + \|u_n(\theta_n(t)) - u_n(t)\|$$
$$\leq L_1 |\theta_n(t) - t| + L_2 \|u_n(\delta_n(t)) - u_n(t)\| + \|u_n(\theta_n(t)) - u_n(t)\|$$

As $n \to \infty$, the right hand side tends to zero, hence $u(t) \in \Gamma(t, u(t))$.

Step5. In this step we show that the sequence $(\tau(\delta_n(t)\, u_n)_{n \geq n_0}$ converges to $\tau(t)u$, for every $t \in I$. Let $t \in I$. We have

$$\| \tau(\delta_n(t))u_n - \tau(t)u \|_{C_H([-r,0])}$$
$$\leq \| \tau(\delta_n(t))u_n - \tau(t)u_n \|_{C_H([-r,0])} + \| \tau(t)u_n - \tau(t)u \|_{C_H([-r,0])}$$
$$\leq \sup_{-r \leq s \leq 0} \|u_n(\delta_n(t) + s) - u_n(t + s)\| + \| \tau(t)u_n - \tau(t)u \|_{C_H([-r,0])}$$
$$\leq \sup_{\substack{-r \leq s_1 \leq s_2 \leq 1 \\ |s_1 - s_2| \leq \frac{1}{n}}} \|u_n(s_1) - u_n(s_2)\| + \| \tau(t)u_n - \tau(t)u \|_{C_H([-r,0])}$$
$$\leq \sup_{\substack{-r \leq s_1 \leq s_2 \leq 0 \\ |s_1 - s_2| \leq \frac{1}{n}}} \|u_n(s_1) - u_n(s_2)\| + \sup_{\substack{-r \leq s_1 \leq 0 \leq s_2 \leq 1 \\ |s_1 - s_2| \leq \frac{1}{n}}} \|u_n(s_1) - u_n(s_2)\|$$
$$+ \sup_{\substack{0 \leq s_1 \leq s_2 \leq 1 \\ |s_1 - s_2| \leq \frac{1}{n}}} \|u_n(s_1) - u_n(s_2)\| + \| \tau(t)u_n - \tau(t)u \|_{C_H([-r,0])}$$
$$\leq \sup_{\substack{-r \leq s_1 \leq s_2 \leq 0 \\ |s_1 - s_2| \leq \frac{1}{n}}} \| \Psi(s_1) - \Psi(s_2) \| + \sup_{\substack{-r \leq s_1 \leq 0 \\ |s_1| \leq \frac{1}{n}}} \|u_n(s_1) - u_n(0)\| +$$
$$+ \sup_{0 \leq s_2 \leq 1} \|u_n(s_1) - u_n(0)\|$$
$$+ \sup_{\substack{0 \leq s_1 \leq s_2 \leq 1 \\ |s_1 - s_2| \leq \frac{1}{n}}} \|u_n(s_1) - u_n(s_2)\| + \| \tau(t)u_n - \tau(t)u \|_{C_H([-r,0])}$$
$$\leq 2 \sup_{\substack{-r \leq s_1 \leq s_2 \leq 0 \\ |s_1 - s_2| \leq \frac{1}{n}}} \| \Psi(s_1) - \Psi(s_2) \| + 2 \sup_{\substack{0 \leq s_1 \leq s_2 \leq 1 \\ |s_1 - s_2| \leq \frac{1}{n}}} \|u_n(s_1) - u_n(s_2)\|$$
$$+ \| \tau(t)u_n - \tau(t)u \|_{C_H([-r,0])}.$$

By the continuity of Ψ, the uniform convergence of u_n towards u and the preceding estimate, we get $\lim_{n \to \infty} | \tau(\delta_n(t)\, u_n - \tau(t)u | = 0$.

Step 6. We show that the sequence $(f_n)_{n \geq n_0}$, defined by $f_n(t) = g_n(\delta_n(t)$, $t \in I$ converges almost everywhere to a function $f \in L^1(I, H)$ and $f(t) \in F(t, \tau(t)u)$, a. e. $t \in I$.To prove this, let $n \geq n_0$ and $t \in I$ be a fixed point such that (23)(iii), (iv) are satisfied and the function m is finite almost everywhere. In view of (H4) we have

$$\| g_{n+1}(t) - g_n(t) \| = d(g_n(t), F(t, \tau(\delta_{n+1}(t))u_{n+1})) + \frac{1}{(n+1)^2}$$
$$\leq h(F(t, \tau(\delta_n)u_n), F(t, \tau(\delta_{n+1}(t)u_{n+1})) + \frac{1}{(n+1)^2}$$
$$\leq m(t) \| \tau(\delta_{n+1}(t))u_{n+1} - \tau(\delta_n)u_n \| + \frac{1}{(n+1)^2}.$$

Thus, for any two natural numbers $n, q\ (n_0 < n < q)$ we infer that

$$\| g_n(t) - g_q(t) \| = \| g_n(t) - g_{n+1}(t) \| + \| g_{n+1}(t) - g_{n+2}(t) \| + \ldots + \| g_{q-1}(t) - g_q(t) \|$$
$$\leq m(t)[\| \tau(\delta_{n+1}(t))u_{n+1} - \tau(\delta_n(t))u_n \| + \| \tau(\delta_{n+2}(t))u_{n+2} - \tau(\delta_{n+1}(t))u_{n+1} \|$$
$$+ \ldots + \| \tau(\delta_q(t))u_q - \tau(\delta_{q-1}(t))u_{q-1} \|] + \frac{q-n}{(n+1)^2}.$$

Since $\lim_{n\to\infty}\delta_n(t)=t$, m is finite and $(\tau(\delta_n(t)\ u_n)_{n\geq n_0}$ converges uniformly to $\tau(t)u$, then the right hand side of the last inequality tends to zero when. $n,q\to\infty$ Hence, for almost $t\in I$, the sequence $(g_n(t)\ _{n\geq 1}$ is a Cauchy sequence in H. Thus there is a function $f:I\to H$ such that $\lim_{n\to\infty}g_n(t)=f(t)$, a. e. $t\in I$.

Observe that, by (H5) we have for almost $t\in I$,

$$\|g_n(t)\|\leq L_3(1+\tau(\delta_n(t))u_n(0))=L_3(1+u_n(\delta_n(t)))\leq L_3(1+\|\Psi(0)\|+\gamma_2T).$$

Then, $f\in L^1(I,H)$. It remains to show that $f(t)\in F(t,\tau(t)u)$, for a.e. $t\in I$. Indeed, for every $n\geq n_0$ and $t\in I$, we have by(H5)

$$\begin{aligned}d(f(t),F(t,\tau(t)u))&\leq\|f(t)-g_n(t)\|+d(g_n(t),F(t,\tau(t)u))\\&\leq\|f(t)-g_n(t)\|+h(F(t,\tau(\delta_n(t))u_n),F(t,\tau(t)u)))\\&\leq\|f(t)-g_n(t)\|+m(t)\|\tau(\delta_n(t))u_n-\tau(t)u\|.\end{aligned}$$

Since g_n converges to f almost everywhere and since m is finite almost everywhere, then by step 5, we conclude that $f(t)\in F(t,\tau(t)u)$, a.e. $t\in I$.

Step 7. In this step we show that: $-u'(t)-f(t)\in N_{\Gamma(t,u(t))}(u(t)),\ a.e.\ t\in I.$

We apply the technique used in (Castaing, et al. 2009). Using the facts that $(u'_n)_{n\geq n_0}$ Converges weakly to u' and $f_n(t)$ converges almost everywhere to *f*, and by Mazur's Lemma, we get

$$-u'(t)-f(t)\in\bigcap_{n\geq n_0}\overline{Co}\{-(u'_k(t)-f_k(t):k\geq n\},\tag{30}$$

For almost $t\in I$. Let $t\in I$ such that (30) is satisfied and $v\in H$ be a fixed point. So

$$<v,-u'(t)-f(t)\leq\inf_{n\geq 2}\sup_{k\geq n}<v,-u'_k(t)-f_k(t)>.$$

From this inequality with (5)(v) and Lemma 1, one obtains

$$<v,-u'(t)-f(t)>\leq\limsup_{n\to\infty}\ \sigma<v,\gamma_1\partial^P d_{\Gamma(\theta_n(t),u_n(\delta_n(t)))}(u_n(\theta_n(t)))>.\tag{31}$$

Next, let us show that Γ is continuous. Indeed, for any (t_1,x_1) $(t_2,x_2)\in I\times H$, and any $z\in\Gamma(t_1,x_1)$ we have

$$d_{\Gamma(t_2,x_2)}(z)=|d_{\Gamma(t_2,x_2)}(z)-d_{\Gamma(t_1,x_1)}(z)|\leq L_1|t_2-t_1|+L_2|\ x_2-x_1|.$$

This means that

$$z\in\Gamma(t_2,x_2)+(L_1|t_2-t_1|+L_2|\ x_2-x_1\|)B.$$

Hence,

$$\Gamma(t_1,x_1)\subseteq\Gamma(t_2,x_2)+(L_1|t_2-t_1|+L_2|\ x_2-x_1\|)B$$

By interchanging the role of $\Gamma(t_1,x_1)$ and $\Gamma(t_2,x_2)$, we conclude that Γ is Hausdorff continuous. Therefore, in virtue of Lemma 2, the relation (31) yields

$$<v,-u'(t)-f(t)>\leq\sigma<v,\gamma_1\partial^P d_{\Gamma(t,u(t))}(u(t))>.$$

Since $u(t)\in\Gamma(t,u(t)$, the set $\partial^P d_{\Gamma(t,u(t)}(u(t)$ is closed convex, hence we obtain

$$-u'(t)-f(t)\in\gamma_1\partial^P d_{\Gamma(t,u(t))}(u(t))\subseteq N^P_{\Gamma(t,u(t))}(u(t)).$$

Again, since $u(t)\in\Gamma(t,u(t)$, $\forall t\in I$, then $N^P_{\Gamma(t,u(t)}(u(t)=N_{\Gamma(t,u(t)}(u(t)$, $\forall t\in I$. Thus,

$$-u'(t)\in N_{\Gamma(t,u(t)}(u'(t)+f(t)\ a.e.\ t\in I.$$

2.Existence result of solutions for the problem(3).

Theorem 2. Let H be a real separable Hilbert space, $K:I\times H\to 2^H$ be a multifunction with nonempty closed uniformly ρ-prox-regular values and $F:I\times C_r\to 2^H$ be a multifunction with nonempty closed values. In addition to the conditions (H1),(H4) and (H5) we assume the following conditions:

(H6) There are two constants $\beta_1>0$ and $\beta_2>0$ such that for all $t,s\in I$ and $x,u,v\in H$

$$|d_{K(t,u)}(x)-d_{K(s,v)}(x)|\leq\beta_1|t-s|+\beta_2|\ u-v|.$$

(H7) For any $t\in I$ and any bounded subset A in H with $\chi(A)>0$ and any $r>0$ one has. $\chi(K(t,A))\bigcap rB)<\chi(A)$

Then, for each $\Psi,\Phi\in C_r$ with $\Psi(0)\in C(0,\Phi(0))$, there are two continuous functions $u,p:[-r,T]\to H$ such that u and p are absolutely continuous on I and (3) is satisfied.

Proof. Let $n_0 \geq 2$ be such that

$(u_n)_{n\geq n_0} \subseteq C([-r,T],H)$ $(p_n)_{n\geq n_0} \subseteq C([-r,T],H)$

Using the same lines of first two steps in the proof of Theorem1, there are three sequences

$$(\beta_1 + \beta_2 k + L_3(1+\|\Phi(0)\|+kT))\frac{T}{2^{n_0}} \leq \frac{\rho}{2}. \tag{32}$$

,

and $(g_n)_{n\geq n_0} \subseteq L^1(I,H)$ such that:

$p_n(t) = \Phi(t)$ and $u_n(t) = \Psi(t)$, for $t \in [-r,0]$

and for $t \in [t_i^n, t_{i+1}^n]$, $i = 0,1,2,...,2^n - 1$,

$$\begin{cases} p_n(t) = p_n(t_i^n) + (t - t_i^n)x_i^n, & (33)\\ u_n(t) = x_i^n + \dfrac{t - t_i^n}{\mu_n}[x_{i+1}^n - (x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds)] - \int_{t_i^n}^{t} g_i^n(s)ds, & (34)\end{cases}$$

where

$$x_0^n = \Psi(0),\ x_{i+1}^n = proj_{\Gamma(t_{i+1}^n, p_n(t_{i+1}^n))}(x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds), \tag{35}$$

$$g_i^n \in L^1([t_i^n, t_{i+1}^n], H), g_i^n(t) \in F(t, \tau(\delta_n(t))p_n),\ \text{a.e. } t \in [t_i^n, t_{i+1}^n], \tag{36}$$

and

$$g_n(t) = g_i^n(t),\ t \in [t_i^n, t_{i+1}^n),\ \mathrm{g_n}(T) = g_{2^n}^n(T). \tag{37}$$

$$\|g_{n+1}(t) - g_n(t)\| \leq d(g_n(t), F(t, \tau(\delta_{n+1}(t))p_{n+1}) + \frac{1}{n^2},\ \text{a.e. } t \in I. \tag{38}$$

Let us show that x_{i+1}^n, given by (35), is well define. By construction one obtains for $i = 0,1,2,...,2^n - 1$

$$\begin{aligned} & d_{K(t_{i+1}^n, p_n(t_{i+1}^n))}(x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds) \\ & \leq d_{K(t_{i+1}^n, p_n(t_{i+1}^n))} x_i^n + \int_{t_i^n}^{t_{i+1}^n} \|g_i^n(s)\|ds \qquad (39) \\ & = |d_{K(t_{i+1}^n, p_n(t_{i+1}^n))}(x_i^n) - d_{K(t_i^n, p_n(t_i^n))}(x_i^n)| + \mu_n L_3(1+\|\tau(t_i^n)p_n(0)\|) \\ & \leq \beta_1\mu_n + \beta_2\|p_n(t_{i+1}^n) - p_n(t_i^n)\| + \mu_n L_3(1+\|\tau(t_i^n)p_n(0)\|) \\ & \leq \beta_1\mu_n + \beta_2\mu_n\|x_i^n\| + \mu_n L_3(1+\|p_n(t_i^n)\|). \end{aligned}$$

Moreover,

$$\begin{aligned} \|p_n(t_i^n)\| &= \|p_n(t_{i-1}^n) + (t - t_{i-1}^n)x_{i-1}^n\| \leq \|p_n(t_{i-2}^n) + (t_{i-1}^n - t_{i-2}^n)x_{i-2}^n\| + \|(t - t_{i-1}^n)x_{i-1}^n\| \\ &\leq \|\Phi(0)\| + \|(t_1^n - t_0^n)x_0^n) + (t_2^n - t_1^n)x_1^n + ... + (t - t_{i-1}^n)x_{i-1}^n\| \\ &\leq \|\Phi(0)\| + kT \end{aligned} \tag{40}$$

Therefore,

$$\begin{aligned} & d_{K(t_{i+1}^n, p_n(t_{i+1}^n))}(x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds) \\ & \leq \beta_1\mu_n + \beta_2\mu_n k + \mu_n L_3(1+\|\Phi(0)\|+kT) \\ & = \mu_n[\beta_1 + \beta_2 k + L_3(1+\|\Phi(0)\|+kT)]. \end{aligned} \tag{41}$$

So,

$$\begin{aligned} & d_{K(t_{i+1}^n, p_n(t_{i+1}^n))}(x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds) \\ & \leq (\beta_1 + \beta_2 k + L_3(1+\|\phi(0)\|+kT))\frac{T}{2^{n_0}} \leq \frac{\rho}{2} < \rho. \end{aligned} \tag{42}$$

Thus, the uniformly ρ–prox-regularity of the values of K and Lemma1, ensure the existence and uniqueness of x_{i+1}^n.

Now, let us show that there is η_1 such that for any $n \geq n_0$, $\| u_n'(t) - g_n(t)| \leq \eta_1$, a.e. Let $i \in \{0,1,2,...,2^n - 1\}$ and $n \geq n_0$ be fixed. By (34), (37) and (41) for almost $t \in [t_i^n, t_{i+1}^n]$,

$$\begin{aligned} \|u_n'(t) + g_i^n(t)\| &= \frac{1}{\mu_n}\|x_{i+1}^n - (x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds))\| \\ &\leq \|\frac{1}{\mu_n}\|x_{i+1}^n - (x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds)\| \\ &\leq \frac{1}{\mu_n} d_{K(t_{i+1}^n, p_n(t_{i+1}^n))}(x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds) \\ &\leq \beta_1 + \beta_2 k + L_3(1+\|\Phi(0)\|+kT) := \eta_1. \end{aligned} \tag{43}$$

Moreover, by arguing as in (25) we can show that $u_n, n \geq n_0$ is absolutely continuous and $\| u_n'(t)| \leq \eta_2$, for almost $t \in I$, where.

$$\eta_2 = \beta_1 + \beta_2 k + L_3(1+\|\varphi(0)\|+kT) + L_3(1+k)$$

Observe that, by the construction, for all $n \geq n_0$ and for almost $t \in I$

$$-u_n'(t) - g_n(t) \in N_{K(\theta_n(t), p_n(\theta_n(t)))}(u_n(\delta_n(t)) \cap \eta_1 B,\ \text{a. e. } t \in I. \tag{44}$$

Next, Let $t \in I$. Then, $\forall n \geq n_0$ there is $i = 0,1,2,...,2^n - 1$ such that $t \in [t_i^n, t_{i+1}^n]$. Hence,

$$\begin{aligned} p_n(t) &= p_n(t_{i-1}^n) + (t - t_{i-1}^n)u_n(t_{i-1}^n) = p_n(t_{i-2}^n) + (t_{i-1}^n - t_{i-2}^n)u_n(t_{i-2}^n) + (t - t_{i-1}^n)u_n(t_{i-1}^n) \\ &= \Phi(0) + (t_1^n - t_0^n)u_n(t_0^n) + (t_2^n - t_1^n)u_n(t_1^n) + ... + (t_{i-1}^n - t_{i-2}^n)u_n(t_{i-2}^n) + (t - t_{i-1}^n)u_n(t_{i-1}^n) \\ &= \Phi(0) + \int_{t_0^n}^{t_1^n} u_n(\delta_n(s))ds + \int_{t_1^n}^{t_2^n} u_n(\delta_n(s))ds + ... + \int_{t_{i-2}^n}^{t_{i-1}^n} u_n(\delta_n(s))ds + \int_{t_{i-1}^n}^{t} u_n(\delta_n(s))ds \\ &= \Phi(0) + \int_0^t u_n(\delta_n(s))ds. \end{aligned} \tag{45}$$

Let us divide the rest of the proof into steps.

Step 1. In this step, we prove that the sequence $(u_n|_I)_{n\geq n_0}$ has a convergent subsequence, still denoted by $(u_n|_I)_{n\geq n_0}$, converging uniformly to a Lipschitz function $u \in C(I,H)$. In fact, since $\|u_n'(t)\| \leq \eta_2$, for almost $t \in I$, the set $\{u_n|_I : n \geq n_0\}$ is equicontinuous. Let $t \in I$

be a fixed point. From (H6), one obtains for every $n \geq n_0$,

$$K(\theta_n(t), p_n(\theta_n(t)))$$
$$\subseteq K(t, p_n(t)) + (\beta_1 |\theta_n(t) - t| + \beta_2 \| p_n(\theta_n(t)) - p_n(t) \|)B$$
$$\subseteq K(t, p_n(t)) + \beta_1 \mu_n B$$

This relation together the fact that $u_n(\theta_n(t) \in K(\theta_n(t), p_n(\theta_n(t)))$, $t \in I$ yields

$$u_n(t) \in (K(t, p_n(t)) \cap RB) + \mu_n(\beta_1 + k + \eta_2 B + k)B, \forall n \geq n_0, t \in I, \quad (46)$$

where $R = \| \Psi(0) \| + T\eta_2$. Let $\varepsilon > 0$.
Since $\lim_{n \to \infty} \mu_n = 0$, we can find a natural number $N_0 \geq n_0$ such that

$\mu_n < \dfrac{\varepsilon}{2(\beta_1 + k + \eta_2)}$, $\forall n \geq N_0$. Then by (46), $u_n(t) \subseteq (K(t, p_n(t)) \cap RB) + \varepsilon/2 B$ $\forall n \geq n_0$. Hence, by (H7)

$$\chi\{u_n(t) : n \geq N_0\} \leq \chi(K(t, \{p_n(t) : n \geq N_0\} \cap RB) + \varepsilon/2 B$$
$$\leq \chi(p_n(t) : n \geq N_0\} + \varepsilon/2$$
$$\leq \chi\{\int_0^t u_n(s)ds : n \geq N_0\} + \varepsilon/2 \leq 2\int_0^t \chi\{u_n(s)ds : n \geq N_0\} + \varepsilon/2. \quad (47)$$

Since the function $t \to \chi\{u_n(t) : n \geq N_0\}$ is continuous, the relation (47) and the usual Gronwall's inequality imply $\chi\{u_n(t) : n \geq N_0\} \leq (\varepsilon/2)\exp(2t)$.
Since ε is arbitrary, we conclude that $\chi\{u_n(t) : n \geq N_0\} = 0$. This means that the set $\{u_n(t) : n \geq N_0\}$ is relatively compact in $C(I, E)$. By applying Th. 4, Ch. 1 in (Aubin, and Cellina,1984) there is a Lipschitz function $u : I \to H$, such that the sequence $(u_n|_I)_{n \geq n_0}$ has a subsequence, still denoted by $(u_n|_I)_{n \geq n_0}$, which converges uniformly to u. We extend the definition of u on $[-r, T]$ by putting $u(t) = \psi(t)$, $[-r, 0]$. Thus $(u_n)_{n \geq 2}$ converging uniformly to u on $[-r, T]$. In addition, by (4.13), the sequence $(u'_n)_{n \geq n_0}$ is uniformly bounded in $L^2(I, H)$. Hence, without loss of generality, we may suppose that there is $z \in L^2(I, H)$ such that $u'_n \to z$ weakly in $L^2(I, H)$. Then, for each $t \in I$,

$$u(t) = \lim_{n \to \infty} u_n(t) = \Psi(0) + \lim_{n \to \infty} \int_0^t u'_n(s)ds = \Psi(0) + \int_0^t z(s)ds$$
.

Thus $u' = z$, a.e. so, u'_n converges weakly to u'.

Step 2. Let us define $p : [-r, T] \to H$ as $p(t) = \Phi(t)$, for $[-r, 0]$ and for $t \in I$,

$$p(t) = \Phi(0) + \int_0^t u_n(\delta_n(s))ds$$

Since, (u_n) converges uniformly to u on I, then (p_n) converges uniformly to p on I.

Step 3. Our aim in this step is to prove that: $u(t) \in K(t, p(t)$, $\forall t \in I$. Let $t \in I$. By (H2)

$$d_{K(t,p(t))}(u(t)) \leq d_{K(t,p(t))}(u_n(\theta_n(t)) + \| (u_n(\theta_n(t)) - u_n(t) \|$$
$$= | d_{K(t,p(t))}(u_n(\theta_n(t)) - d_{K(\theta_n(t), p_n(\theta_n(t)))}(u_n(\theta_n(t)) |$$
$$+ \| (u_n(\theta_n(t)) - u_n(t) \|$$
$$\leq \beta_1 \mu_n + \beta_2 \| p_n(\theta_n(t)) - p_n(t) \| + \| u_n(\theta_n(t)) - u_n(t) \|.$$

As the set $K(t, p(t)$ is closed and the right hand side tends to zero when $n \to \infty$, one obtains $u(t) \in K(t, p(t)$.

Step 4. By arguing as in steps 5-7 in the proof of Theorem 1, we can show that the sequence $(\tau(\delta_n(t) p_n)_{n \geq n_0}$ converges to $\tau(\delta(t) p$, for every $t \in I$, the sequence $(g_n)_{n \geq n_0}$ converges almost everywhere to a function $f \in L^1(I, H)$ with $f(t) \in F(t, \tau(t)p)$, a.e. $t \in I$ and $-u'(t) - f(t) \in N_{K(t,p(t))}(u(t)), a.e.\ t \in I.$
This completes the proof.

DISCUSSION

In this paper, existence results of solution of functional sweeping processes of first and second order in Hilbert space with noncompact perturbation have been established. Some sufficient conditions have been obtained. The importance of this work is the values of the perturbation are not necessarily compact. Moreover, the values of the moving set are not contained in a fixed compact set. We would like to refer that, in order to show that the sequence (u_n) has a convergent subsequence, (Castaing *et al*., 2009) assumed the condition: For any bounded and for any convergent sequence (t_n) in $I=[0,T]$ and for any bounded set $A\subseteq H$ the set $\bigcup\{K(t_n,x): n\geq 1, x\in A\}$ is ball compact. It is easy to see that this condition implies (H_7).

ACKNOLEDGEMENT

The authors gratefully acknowledge the Deanship of Scientific Research, King Faisal University, for their financial support this research project No.150153.

REFERENCES

Aitalioubrahim, M., 2012. On noncompact perturbation of nonconvex sweeping process. Comment. Math. Univ. Carolina. 53(1): 65-77.

Aubin, J. P., and Cellina, A. 1984. Differential Inclusions: Set-Valued Maps and Viability Theory. Springer-Verlag, Berlin.

Castaing, C., Ibrahim, A. G., and Yarou, M. 2009. Some contributions to nonconvex sweeping process. Journal of Nonlinear and Convex Analysis. 10(1): 1-20.

Chemetov, N., and Monteiro Marques, M. D. P. 2007. Non-convex quasi-variationaldifferential inclusions. Set-valued Anal. (15): 209-221.

Edmond, J. F., and Thibault, L. 2006. BV solutions of non-convex sweeping process differential inclusions with perturbations. J. Differential Equations. 226: 133-179.

Gomaa, A., 2013. Relaxation problems involving second order differential inclusions. Abstr. Appl. Anal. 2013: Article ID 792431.

Haddad, Touma, and Haddad, Tahar. 2014. Existence of solutions to the state dependent sweeping process with delay. J. Nonlinear Sci. Appl. 7:70-77.

Haddad, T. 2013. Nonconvex differential variational inequality and state-dependent sweeping process. J. Optim Theory Appl. Doi10.1007/s 10957 013-0353-1

Moreau, J. J., 1977. Evolution problem associated with a moving convex set in a Hilbert space. J. Differential Equations. 26: 347-374.

Noel, J., and Thibault, L. 2014. Nonconvex sweeping process with a moving set depending on the state. Vietnam J. Math. 42:595-612.

Thibault, L. 2003. Sweeping process with regular and non-regular sets. J. Differential Equations. 193:1-26.

Zhu, Q. J. 1991. On the solution set of differential inclusions in Banach space. J. Differential Equations. 93: 213-237.

عملية كسح غير محدبة باضطراب غير متراص مع تأخير

أحمد جمال إبراهيم و فريال عبد الله العدساني

قسم الرياضيات، كلية العلوم، جامعة الملك فيصل
الأحساء، المملكة العربية السعودية

استلام 25 أكتوبر 2015م - قبول 29 مايو 2016

الملخص

إن عمليات الكسح يقابلها مسائل ميكانيكية مهمة. لذلك فإن العديد من المؤلفين اهتموا بنظريات وجود حلول لمسائل كسح.

هذا البحث يبرهن نتائج وجود حلول تخص احتواءات تفاضلية دالية متولدة من عمليات كسح؛ حيث تكون المجموعة المتحركة دالة ذات قيم مجموعية مغلقة لكنها ليست بالضرورة محدبة وتعتمد على متغيرين مع وجود اضطراب ذي قيم ليست بالضرورة متراصة.

واعتمادًا على تقنية مستخدمة في أبحاث حديثة نبرهن نظريتين جديدتين، الأولى تخص الشروط الكافية لضمان وجود حل لاحتواء تفاضلي من الرتبة الأولى ومتولد من عملية كسح حيث تكون المجموعة المتحركة دالة ذات قيم مجموعية مغلقة لكنها ليست بالضرورة محدبة وتعتمد على متغيرين مع وجود اضطراب ذي قيم ليست بالضرورة متراصة. أما في الثانية فنعطي الشروط الكافية لضمان وجود حل لاحتواء تفاضلي من الرتبة الثانية ومتولد من عملية كسح حيث تكون المجموعة المتحركة دالة ذي قيم مجموعية مغلقة لكنها ليست بالضرورة محدبة وتعتمد على متغيرين مع وجود اضطراب ذي قيم ليست بالضرورة متراصة. لن نفرض أن قيم المجموعة المتحركة محتواة في مجموعة متراصة ثابتة كما في معظم الأبحاث، وبدلا من ذلك سنفرض شرطًا يحتوي على مقياس عدم التراص.

علاوة على ذلك فإن التقنية المستخدمة في هذا البحث تسمح بدراسة مسائل عمليات كسح باضطراب وقيمة غير متراصة.

الكلمات المفتاحية: الاحتواء التفاضلي، عمليات الكسح، المخروط العمودي.