Comments on the Transient Solution of Stochastic Fluid Models

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Received 31 March 2015 - Accepted 13 November 2015

ABSTRACT

The joint probability distribution of the buffer content and the state of the Markov process controlling the input/output rates in a stochastic fluid model has been widely investigated. This probability distribution in transient regime fulfills a partial linear differential system and many techniques of resolution in the literature have been proposed. But the uniqueness of the solution has been never studied. In this paper, the boundary conditions to ensure uniqueness were precisely specified. The jumps with specific values in some critical points are necessary to guarantee the uniqueness of the solution. This work showed also that some methods using the Laplace transform do not take into account of the discontinuity property leading in that way to different mathematical mistakes depending on the used approach. Finally, this paper showed that the approach using the uniformization technique gives the exact solution and its proof is completed.

Key Words: Laplace transform, Markov process, Partial differential equations, Stochastic fluid models.

INTRODUCTION

Many researchers are interested in the Markov modulated fluid models, especially in the numerical computation of the cumulative distribution function of the fluid level in a storage device. This distribution function is governed by a partial differential system. Different approaches have been developed in literature to solve this system of partial differential equations. In this context, the Laplace transform is widely used (Kobayashi and Ren (1992), Ren and Kobayashi (1995), Tanaka *et al.* (1995), Ahn and Ramasmawi (2004), Saghouani and Mandjes (2011)). All these methods did not take into account the discontinuity property of the solution. By omitting this property, different mathematical mistakes are induced depending on the used approach. All these mathematical remarks will be explained in Section 5. Sericola (1998) adopted the uniformization technique to have a numerically stable solution. He was inspired from the performability solution developed by Nabli (1995) and applied to a fault tolerant multiprocessor (Nabli and Sericola (1994) and (1996)) to give a similar solution for this system. Nevertheless, the boundary conditions specified in his proof are insufficient to get the uniqueness. In this paper, all boundary conditions needed to achieve uniqueness were precisely described and the crucial property of discontinuity was shown. Finally, the proof of Sericola (1998) was completed to conclude that his solution is actually the unique solution of the problem.

MATERIALS AND METHODS

Mathematical model

Stochastic fluid model has been widely investigated in the two last decades. This model is used to describe the random behavior of the fluid level in a buffer when the input ρ_i and output c_i are controlled by a homogeneous Markov process $(X_t)_{t\geq 0}$ on a finite state space S. The aim is to compute, by a numerically stable method, the distribution function $x \mapsto \mathsf{P}(Q_t > x)$ of the buffer content Q_t for all time $t > 0$. At time 0, the buffer is supposed empty, that is $Q_0 = 0$. It is well-known that the joint probability distribution $F_i(t,x) = \mathsf{P}(Q_t > x, X_t = i)$ is governed by the following partial differential equations (see for instance Tanaka *et al.* (1995) and/or Da Silva Soares (2005)):

$$\frac{\partial F_i(t,x)}{\partial t} = -d_i \frac{\partial F_i(t,x)}{\partial x} + \sum_{l \in S} F_l(t,x) A_{li}, \forall i \in S \quad ... (1)$$

where A is the infinitesimal generator of the Markov process $(X_t)_{t\geq 0}$ and $d_i = \rho_i - c_i$ designates the effective input rate associated to state *i*. Since two states may have the same effective input rate, the different values of d_i ; $i \in S$; are denoted by $r_0, r_1, \ldots, r_m$, where m is naturally an integer less than the cardinality of *S*. These distinct values are ordered in ascending order:

$$r_m > r_{m-1} > \cdots > r_u > 0 \geq r_{u-1} > \cdots > r_1 > r_0.$$

The state space S of the Markov process $(X_t)_{t\geq 0}$ can be partitioned into $m+1$ disjoint subsets $B_m, \ldots, B_0$, where B_i is the set composed by all states having the same effective input rate r_i. For commodity reason, let S^+ denotes the subset of all states having positive effective input rate and S^- denotes its complementary. The natural boundary conditions commonly used in the literature are as follow:

$$\begin{cases} F_i(t,0) = \mathsf{P}(X_t = i), & \text{for } i \in S^+ \\ F_i(t,x) = 0, \forall x \geq r_m t, & \text{for } i \in S \\ F_i(0,x) = 0, \forall x \geq 0, & \text{for } i \in S. \end{cases}$$

... (2)

The first condition states that the buffer content Q_t is certainly nonempty when the process is in a filling state. The second condition is due to the fact that Q_t is upper bounded by $r_m t$, and the last one explains the hypothesis $Q_0 = 0$. The first two conditions cover the variable x, that corresponds in reality to the amount of fluid, and the last one covers the time t. It will be seen in the next section that these boundary conditions are incomplete to have the exact solution. All methods of Kobayashi and Ren (1992), Ren and Kobayashi (1995),Tanaka *et al.* (1995), Ahn and Ramasmawi (2004), Saghouani and Mandjes (2011), which use the Laplace transform, supposed implicitly that the function $x \mapsto F_i(t,x)$ is of class C^1. However, this property is not true. The author who includes the discontinuity points and their jump values was Sericola (1998). Nevertheless, his proof is incomplete since it does not show that the proposed solution has really a jump on each critical point. Without this last condition, we will show that the problem has multiple solutions.

Recalling the solution proposed by Sericola (1998) for the joint distribution of the bivariate (Q_t, X_t), this solution was similar to the one proposed by Nabli and Sericola (1994), and Nabli (1995) for performability measure. It is interesting to state that the latter was obtained directly from an approach based on integral equations and not through a partial differential system as done in this study.

RESULTS AND DISCUSSION

Theorem 2.1

For all state $i \in S$, we have:

- $\forall x \in [0, r_u t)$

$$F_i(t,x) = \sum_{n\geq 0} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{r_u t}\right)^k \left(1 - \frac{x}{r_u t}\right)^{n-k} b_i^{(u)}(n,k)$$

- $\forall j = u+1, \ldots, m$, $\forall x \in [r_{j-1} t, r_j t)$

$$F_i(t,x) = \sum_{n\geq 0} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} (x_j)^k (1 - x_j)^{n-k} b_i^{(j)}(n,k)$$

where $x_j = \dfrac{x - r_{j-1}t}{(r_j - r_{j-1})t}$. *The coefficient λ is the uniformization rate of the Markov process $(X_t)_{t\geq 0}$ and $b_i^{(j)}(n,k)$; $u \leq j \leq m$; are real sequences, computed iteratively as a convex combination of two elements in the interval [0,1].*

In the expressions of $F_i(t,x)$ mentioned above, the convention of 0^0 (which may happens if $x = r_{j-1}t$ and $k = 0$) is $0^0 = 1$.

Uniqueness

In this section, the boundary conditions of the differential system (1) were fixed. As explained above, the discontinuity points of the function $x \mapsto F_i(t,x)$ will be specified with their jump values. It is also proved that these conditions are crucial for the solution uniqueness.

The amount of fluid Q_t may be at level $r_j t$ if the driver process $(X_t)_{t\geq 0}$ remains in the subset B_j during the observation period [*0, t*]. This ascertainment is mathematically expressed by:

$$\mathsf{P}(Q_t = r_j t, X_t = i) = \alpha_{B_j} e^{A_{B_j B_j} t} e_i,$$

where α_{B_j} (*resp.* $A_{B_j B_j}$) is the sub-vector of

$\boldsymbol{\alpha}$e initial distribution $\boldsymbol{\alpha} := (\mathsf{P}(X_0 = i), i \in S)$ (*resp.* submatrix of A) related to B_j and e_i is the *i*th vector of the canonical basis of $\mathbf{R}^{|B_j|}$. So, for all $i \in B_j$, the function $x \mapsto F_i(t,x)$ has a jump at point $x = r_j t$, which is equal to $\alpha_{B_j} e^{A_{B_j B_j} t} e_i$. On the other hand, starting from a state $i \notin B_j$, the amount of fluid Q_t cannot be at level $r_j t$ (*i.e* $\mathsf{P}(Q_t = r_j t, X_t = i) = 0$, for $i \notin B_j$). Merging the partial differential system (1) with the boundary conditions (2), the whole problem governing the joint distribution of (Q_t, X_t) is:

$$\begin{cases} \dfrac{\partial F_i(t,x)}{\partial t} = -d_i \dfrac{\partial F_i(t,x)}{\partial x} + \sum_{l\in S} F_l(t,x) A_{li}, \forall i \in S, \forall x \in (0, r_m t) \\ F_i(t,0) = \mathsf{P}(X_t = i), \forall i \in S^+ \\ F_i(t,x) = 0, \forall i \in S, \forall x \geq r_m t \\ F_i(0,x) = 0, \forall i \in S, \forall x \in \mathbf{R}_+ \\ \forall j = u,\ldots,m, JF_i(t, r_j t) = 0, \forall i \notin B_j \\ \forall j = u,\ldots,m, JF_i(t, r_j t) = \alpha_{B_j} e^{A_{B_j B_j} t} e_i, \forall i \in B_j \end{cases}$$

... (3)

where $JF_i(t, r_j t)$ stands for the jump of the function $x \mapsto F_i(t,x)$ at point $r_j t$. Since $x \mapsto F_i(t,x)$ is right continuous and decreasing, the jump at each point x_0 is equal to $F_i(t, x_0^-) - F_i(t, x_0)$. Note that $JF_i(t, r_j t)$ coincides exactly with the probability $\mathsf{P}(Q_t = r_j t, X_t = i)$.

The random variable $(Q_t, X_t = i)$ has a probability distribution of mixed type, since $F_i(t,x)$ is discontinuous at point $x = r_j t$ under the hypothesis $\alpha_{B_j} \neq 0_{B_j}$. According to Lebesgue's decomposition theorem, the probability measure of $(Q_t, X_t = i)$ is the sum of an absolutely continuous probability measure and a discrete probability. More precisely, we have:

$$\frac{\partial F_i(t,x)}{\partial x} dx = -f_i(t,x)dx - \alpha_{B_j} e^{A_{B_j B_j} t} e_i \delta_{r_j t}(dx),$$

(4)

where $f_i(t,x)$ is the probability density of $(Q_t, X_t = i)$ and $\delta_{r_j t}$ is the Dirac measure at point $r_j t$. On the other hand, the set in which we look for the vector solution $\mathbf{F}(t,x) = (F_i(t,x), i \in S)$ is $C^1(\Omega, \mathbf{R}^n)$, where n is the cardinality of the state space S and

$$\Omega = \mathbf{R}_+ \times \mathbf{R}_+ \setminus \left(\left(\bigcup_{j=u}^{m} \{(t, r_j t), t \geq 0\}\right) \cup (\mathbf{R}_+ \times \{0\}) \cup (\{0\} \times \mathbf{R}_+) \right).$$

It is clear that Ω is an open set for the usual topology in $\mathbf{R}^2$. The boundary conditions of Problem (3) refer to the border of Ω. Also, the range of $\mathbf{F}(t,x)$ is actually in the interval $[0,1]^n$, since each component $F_i(t,x)$ is a probability.

Theorem 3.1

In $C^1(\Omega, \mathbf{R}^n)$, *the Problem (3) has a unique solution.*

Proof: Suppose Problem (3) has two solutions, the difference, let say $H_i(t,x)$, satisfies the following problem:

$$\begin{cases} \dfrac{\partial H_i(t,x)}{\partial t} = -d_i \dfrac{\partial H_i(t,x)}{\partial x} + \sum_{l\in S} H_l(t,x)A_{li}, \forall i \in S, \forall x \in (0, r_m t) \\ H_i(t,0) = 0, \forall i \in S^+ \\ H_i(t,x) = 0, \forall i \in S, \forall x \geq r_m t \\ x \mapsto H_i(t,x) \text{ is continuous on } \mathbf{R}_+, \forall i \in S \\ H_i(0,x) = 0, \forall i \in S, \forall x \geq 0 \end{cases}$$

... (5)

Let us introduce the Laplace transform $H_i^*(s,x)$ of $H_i(t,x)$ with respect to variable t :

$$H_i^*(s,x) = \int_0^\infty e^{-st} H_i(t,x)dt, \text{ for } s > 0.$$

This Laplace transform is well defined giving that the function $x \mapsto H_i(t,x)$ is continuous and $|H_i(t,x)| \leq 1$ as the difference of two solutions taking their values in [0,1].

For the partial derivative $\dfrac{\partial H_i(t,x)}{\partial x}$, the Dirac quantity appearing in Identity (4) vanishes and it remains only the difference of two probability density, let say $f_i^1(t,x)$ and $f_i^2(t,x)$:

$$\frac{\partial H_i(t,x)}{\partial x} = f_i^1(t,x) - f_i^2(t,x).$$

The boundary condition $H_i(t,x) = 0, \forall i \in S, \forall x \geq r_m t$ leads to the following:

$$\sup_{x>0} \left|\frac{\partial H_i(t,x)}{\partial x}\right| \leq \sup_{x>0} f_i^1(t,x) + \sup_{x>0} f_i^2(t,x) =$$

$$\sup_{x\in(0,r_m t)} f_i^1(t,x) + \sup_{x\in(0,r_m t)} f_i^2(t,x) < \infty$$

In the expression above, it is useful to recall that a probability density function is by definition nonnegative. Since the integral $\int_0^\infty e^{-st}dt$ converges for $s > 0$, thanks to the Leibniz integral rule the permutation between integral and derivative with respect to x is true:

$$\frac{\partial H_i^*(s,x)}{\partial x} = \int_0^\infty e^{-st} \frac{\partial H_i(t,x)}{\partial x} dt, \text{ for } s > 0.$$

By taking into account of the partial differential system and the last condition in (5), one can prove easily that the row vector $\mathbf{H}^*(s,x) = (H_i^*(s,x), i \in S)$ satisfies the following:

$$s\mathbf{H}^*(s,x) = -\frac{\partial \mathbf{H}^*(s,x)}{\partial x} D + \mathbf{H}^*(s,x)A$$

$$\Rightarrow \mathbf{H}^*(s,x)(sI - A) = -\frac{\partial \mathbf{H}^*(s,x)}{\partial x} D$$

$$\Rightarrow \mathbf{H}^*(s,x) = \mathbf{H}^*(s,0)e^{-xW(s)}, \text{ where } W(s) = (sI - A)D^{-1}$$

$$\Rightarrow \mathbf{H}^*(s,x) = \left(H_{S^+}^*(s,0) \quad H_{S^-}^*(s,0)\right)e^{-xW(s)}$$

Now, it is needed to show that $\mathbf{H}^*(s,0) = \mathbf{0}$, it will be done in two phases: $H_{S^+}^*(s,0) = 0_{S^+}$ and then $H_{S^-}^*(s,0) = 0_{S^-}$. In fact, the condition "$H_i(t,0) = 0, \forall i \in S^+$" leads to the equality $H_{S^+}^*(s,0) = 0_{S^+}$. For the states $i \in S^-$, we take advantage of the boundary condition "$H_i(t,x) = 0, \forall i \in S, \forall x \geq r_m t$". An immediate consequence of this condition is to state that $\lim_{x\to\infty} H_i^*(s,x) = 0$; for all $i \in S$; therefore the vector $\mathbf{H}^*(s,x)$ is orthogonal to all eigenvectors of $W(s)$ associated to eigenvalues with negative real parts. According to Lemma 1 of Tanaka *et al.* (1995), the matrix $W(s)$ has exactly $|S^-|$ eigenvalues with negative real part. Since $\mathbf{H}^*(s,x) = \left(0_{S^+} \quad H_{S^-}^*(s,0)\right)$ and the $|S^-|$ eigenvectors associated to the eigenvector with negative real part are linearly independent, the condition of orthogonality mentioned above implies necessarily $H_{S^-}^*(s,0) = 0_{S^-}$. So, the Laplace transform $\mathbf{H}^*(s,x)$ is null and therefore $\mathbf{H}(t,x) = (H_i(t,x), i \in S)$ is of course null for every fixed x and almost all t. The continuity of the function $x \mapsto H_i(t,x)$

guarantees the nullity of $\mathbf{H}(t,x)$ for all x and t. Thereby, the uniqueness is proved. ■

It is interesting to specify that the stability condition $\sum_{i\in S} d_i \pi_i < 0$, where π_i is the steady-state probability of the process $(X_t)_{t\geq 0}$, has no hand in the proof of uniqueness. This point is predictable since in transient regime, the stability condition is not required at all. In asymptotic regime, the uniqueness solution has been proved by Nabli and Ouerghi (2009) through a spectral analysis of the matrix AD^{-1}.

The discontinuity property of the probability distribution $F_i(t,x)$ is crucial for the solution uniqueness, without it the solution is multiple as it will be shown in the next theorem. If the discontinuity property is removed from the boundary conditions, the problem becomes as follows:

$$\begin{cases} \dfrac{\partial F_i(t,x)}{\partial t} = -d_i \dfrac{\partial F_i(t,x)}{\partial x} + \sum_{l\in S} F_l(t,x) A_{li}, \forall i \in S, \forall x \in (0, r_m t) \\ F_i(t,0) = \mathsf{P}(X_t = i), \forall i \in S^+ \\ F_i(t,x) = 0, \forall i \in S, \forall x \geq r_m t \\ \forall j = u,\dots,m, JF_i(t, r_j t) = 0, \forall i \notin B_j \\ F_i(0,x) = 0, \forall i \in S, \forall x \geq 0. \end{cases}$$

... (6)

The next theorem proposes another solution different from the one given in Theorem 2.1. It will be seen that the number of solutions is infinite in this case. Henceforth, $1\!\mathrm{I}_C$ stands for the indicator function which is equal to 1 if C is true and 0 otherwise.

Theorem 3.2

The functions $G_i(t,x)$; $i \in S$; defined on the interval $[0, r_m t)$, are also solution of Problem(6):

$$G_i(t,x) = \sum_{n\geq 0} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{r_m t}\right)^k \left(1 - \frac{x}{r_m t}\right)^{n-k} w_i(n,k) + \alpha_i 1\!\mathrm{I}_{\{i\in S^+\setminus B_m, x=0\}}$$

The real sequence $(w_i(n,k))_{n\geq k}$ is defined by the following recursive expressions:

- *For $i \in S^+$*

$$w_i(n,k) = \frac{d_i - r_m}{d_i} w_i(n,k-1) + \frac{r_m}{d_i} \sum_{l\in S} w_l(n-1,k-1) P_{li}, \forall k = 1,\dots,n-1$$

and

$$w_i(n,n) = \begin{cases} \sum_{l\in B_m} w_l(n-1,n-1) P_{li}, & \text{for } i \in B_m \\ 0 & \text{for } i \in S^+ \setminus B_m \end{cases}$$

- *For $i \in S^-$*

$$w_i(n,k) = \frac{-d_i}{r_m - d_i} w_i(n,k+1) + \frac{r_m}{r_m - d_i} \sum_{l\in S} w_l(n-1,k) P_{li}, \forall k = n-1,\dots,0$$

The initial conditions are:

$$w_i(n,0) = \begin{cases} \alpha_i & \text{if } n = 0 \ \& \ i \in B_m \\ (\mathbf{\acute{a}} P^n)_i & \text{if } n \geq 1 \ \& \ i \in S^+ \end{cases}$$

and $w_i(n,n) = 0, \forall i \in S^-$,

where $\mathbf{\acute{a}} = (\alpha_i, i \in S) = (\mathsf{P}(X_0 = i), i \in S)$ *is the initial distribution vector and* $P = \frac{A}{\lambda} + I$.

Proof: First, let us show that the functions $G_i(t,x)$ are well-defined. Since the sequence $w_i(n,k)$ is defined as a convex combination of two elements, one can prove by induction that:

$$0 \leq w_i(n,k) \leq (\mathbf{\acute{a}} P^n)_i, \forall i \in S, \forall n \geq 0, \forall k = 0,\dots,n.$$

Then, the series appearing in $G_i(t,x)$ satisfies the following:

$$\begin{aligned} 0 &\leq e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{r_m t}\right)^k \left(1 - \frac{x}{r_m t}\right)^{n-k} w_i(n,k) \\ &\leq e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{r_m t}\right)^k \left(1 - \frac{x}{r_m t}\right)^{n-k}, \text{ since } w_i(n,k) \leq (\mathbf{\acute{a}} P^n)_i \leq 1 \\ &= e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \text{ since } \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{r_m t}\right)^k \left(1 - \frac{x}{r_m t}\right)^{n-k} = 1. \end{aligned}$$

Giving that $\sum_{n\geq 0} e^{-\lambda t} \frac{(\lambda t)^n}{n!} = 1$, the series appearing in $G_i(t,x)$ is absolutely convergent.

To make the derivative computation easier, it is imperative to write $G_i(t,x)$ with a simplified expression:

$$G_i(t,x)=\sum_{n\geq 0}\lambda^n e^{-\lambda t}\sum_{k=0}^{n}\frac{(\frac{x}{r_m})^k(t-\frac{x}{r_m})^{n-k}}{k!(n-k)!}w_i(n,k).$$

So, the derivatives with respect to variables t and x are equal to:

$$\frac{\partial G_i(t,x)}{\partial t}=-\lambda G_i(t,x)+\lambda\sum_{n\geq 0}\lambda^n e^{-\lambda t}\sum_{k=0}^{n}\frac{(\frac{x}{r_m})^k(t-\frac{x}{r_m})^{n-k}}{k!(n-k)!}w_i(n+1,k),$$

and

$$\frac{\partial G_i(t,x)}{\partial x}$$

$$=\lambda\sum_{n\geq 0}\lambda^n e^{-\lambda t}\sum_{k=0}^{n}\frac{(\frac{x}{r_m})^k(t-\frac{x}{r_m})^{n-k}}{k!(n-k)!}\left[\frac{w_i(n+1,k+1)-w_i(n+1,k)}{r_m}\right]$$

Keeping into consideration the uniformization equality $P=\frac{A}{\lambda}+I$, we obtain:

$$\sum_{l\in S}G_l(t,x)A_{li}$$

$$=-\lambda G_i(t,x)+\lambda\sum_{l\in S}G_l(t,x)P_{li}$$

$$=-\lambda G_i(t,x)+\lambda\sum_{n\geq 0}\lambda^n e^{-\lambda t}\sum_{k=0}^{n}\frac{(\frac{x}{r_m})^k(t-\frac{x}{r_m})^{n-k}}{k!(n-k)!}\sum_{l\in S}w_l(n,k)P_{li}$$

In order to satisfy the main partial differential system (1), just take $w_i(n,k)$ such that

$$w_i(n+1,k)=\frac{d_i}{r_m}\left(w_i(n+1,k)-w_i(n+1,k+1)\right)+\sum_{l\in S}w_l(n,k)P_{li}.$$

The equality above can be splitted in two forms, the first concerns $i\in S^+$ and the second form is reserved to state $i\in S^-$. It is easy to check that these two forms coincide exactly with the expressions given in the theorem subject. Remark also that the formula $w_i(n,n)=\sum_{l\in B_m}w_l(n-1,n-1)P_{li}$, related to $i\in B_m$, is the same as the general one since $d_i=r_m$ for $i\in B_m$ and $w_i(n,n)=0$ for $i\notin B_m$.

For the boundary conditions of Problem (6), in accordance with the expression of $G_i(t,x)$, it is clear that this function is continuous with respect to variable x on the open interval $(0,r_m t)$. So, for all $j=u,\ldots,m-1$, the function $x\mapsto G_i(t,x)$ has no jump at point $x=r_j t$, for $i\notin B_j$. For the case $j=m$, by taking into account of condition "$F_i(t,r_m t)=0$, for $i\in S$", the continuity of the function $x\mapsto G_i(t,x)$ for each $i\notin B_m$ is achieved by the condition $w_i(n,n)=0$. Finally, the initial conditions related to $w_i(n,0)$ ensure the boundary condition $G_i(t,0)=\mathrm{P}(X_t=i)$ for $i\in S^+$. For the last condition, the function $x\mapsto G_i(t,x)$ is defined on the interval $[0,r_m t)$ and it is implicitly null for $x\geq r_m t$. When $t=0$, the interval $[0,r_m t)$ is reduced to the empty set, so the condition "$G_i(0,x)=0,\ \forall i\in S,\ \forall x\geq 0$" is well satisfied. The proof of this theorem is therefore achieved.

■

It is clear that $G_i(t,x)$ is completely different from the solution $F_i(t,x)$ proposed in Theorem 2.1. One can check for example that $G_i(t,x)$ is differentiable at point $x=r_u t$ but $F_i(t,x)$ is not. On the other hand, since the solution is a probability distribution and the differential system is linear, each convex combination of two different solutions is also a solution of Problem (6). So, we can conclude that Problem (6) has an infinite number of solutions. In the proof of Theorem 2.1, the author uses in his proof the differential system (6) instead of (3). For more details, see the proof in the appendix of Sericola (1998). To be sure of the solution validity, one must check the discontinuity property, this will be achieved in Section 5.

Useful comments

The moment generating function of

$(Q_t, X_t = i)$, which coincides with $-\int_0^\infty e^{-st} \frac{\partial F_i(t,x)}{\partial x} dt$, must be handled with an extreme care. In fact, because of discontinuity of the function $x \mapsto F_i(t,x)$, the permutation between integral and derivative with respect to x is forbidden:

$$\int_0^\infty e^{-st} \frac{\partial F_i(t,x)}{\partial x} dt \neq \frac{\partial}{\partial x} \int_0^\infty e^{-st} F_i(t,x) dt.$$

For illustrating this claim, we propose the following example:

$$F_i(t,x) = \begin{cases} 1 & \text{if } x < 0 \\ 1 - \dfrac{x}{2t} & \text{if } 0 \le x < t \\ \dfrac{1}{4} e^{-(x-t)} & \text{if } t \le x < 2t \\ 0 & \text{if } x \ge 2t \end{cases}$$

Kobayashi and Ren (1992), Ren and Kobayashi (1995), Tanaka *et al.* (1995) made this mistake by interchanging the derivative and integral operators whereas the integrated function is not continuous. Anyway, all these solutions do not take into account of the jump at point $x = r_j t$.

Ahn and Ramasmawi (2004) constructed, for each time $t \ge 0$, a stochastic process $(Q_t^{(n)})_{n \ge 0}$ of buffer contents that converges in probability to Q_t. As a direct consequence, it is stated in their Theorem 6 page 81, that the equality

$$\lim_{n\to\infty} \mathsf{P}(Q_t^{(n)} > x, X_t = i) = \mathsf{P}(Q_t > x, X_t = i)$$

holds for all real $x \ge 0$. Since the random variable $(Q_t, X_t = i)$ is of mixed type, the above equality holds only for number $x \ge 0$ at which the cumulative distribution function $x \mapsto F_i(t,x) = \mathsf{P}(Q_t > x, X_t = i)$ is continuous. This claim may explain the difference in numerical results compared with the method of Sericola (1998) especially at point $x = 0$, which is actually a discontinuity point (see Table 1, page 95 of Ahn and Ramasmawi (2004)). The uniformization technique is acknowledged to be numerically stable and accurate, so we do conjecture that the solution of Sericola (1998) is the more accurate one, contrary to what is stated by Ahn and Ramasmawi (2004).

Many methods are interested in the determination of the busy period distribution. This random variable is governed by a differential system identical to (1) and its distribution is of mixed type (Barbot *et al.* 2001). The busy period in fluid models is the remaining time until the buffer becomes empty. Mathematically speaking, it is defined by $T = \inf\{t > 0 / Q_t = 0\}$. Saghouani and Mandjes (2011) derive an integral equation for Laplace transform of T, but they do not take into account of its mixed nature. More precisely, if $\zeta(s/x,i)$ stands for the Laplace transform $\mathsf{E}(e^{-sT} / Q_0 = x, X_0 = i)$, according to Saghouani and Mandjes (2011), it fulfills the following identity:

$$\zeta(s/x,i) = \sum_{k\neq i} A_{ik} \int_0^\infty e^{A_{ii}u} e^{-su} \zeta(s/x + r_j u, k) du,$$

where j is the index such that $r_j = \rho(i)$. Since the cumulative probability distribution $t \mapsto \mathsf{P}(T > t / Q_0 = x, X_0 = i)$ has a jump at the point $t = \frac{x}{r_j}$ equal to $e_i e^{A_{B_jB_j}\frac{x}{r_j}} \mathbf{1}$, where $\mathbf{1}$ is a column vector of 1's, the equation above is false and must be as follows:

$$\zeta(s/x,i) = e^{-s\frac{x}{r_j}} e_i e^{A_{B_jB_j}\frac{x}{r_j}} \mathbf{1} + \sum_{k\neq i} A_{ik} \int_0^\infty e^{A_{ii}u} e^{-su} \zeta(s/x + r_j u, k) du.$$

The element e_i was defined, at the beginning of Section 3, as the ith vector of the canonical basis of $\mathbf{R}^{|B_j|}$. Here, e_i is a row vector and previously it was a column vector. The nature of the vector e_i (column or row) will be given by the context.

The solution

The purpose of this section is to show that the solution proposed by Sericola (1998) and recalled in Theorem 2.1 is actually the exact solution of Problem (3). It is sufficient to show that $F_i(t,x)$ satisfies the discontinuity property, because it is the missing property in the proof of Sericola (1998). The following theorem shows that the condition related to the jump at point $x = r_j t$ is really satisfied.

Theorem 5.1

The solution $F_i(t,x)$ *defined in Theorem 2.1 fulfills:*

$$\forall j = u,\dots,m,\ JF_i(t,r_j t) = \alpha_{B_j} e^{A_{B_j B_j} t} e_i,\ \forall i \in B_j .$$

Proof: Let $j \in \{u,\dots,m\}$ and $i \in B_j$, according to the expression of $F_i(t,x)$ in Theorem 2.1, we have:

$$\begin{aligned} JF_i(t,r_j t) &= F_i(t,r_j t^-) - F_i(t,r_j t) \\ &= \sum_{n\geq 0} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\left(b_i^{(j)}(n,n) - b_i^{(j+1)}(n,0)\right). \end{aligned}$$

To be self-containing, we recall here the recursive expressions related to the sequence $b_i^{(j)}(n,k)$:

$$(d_i - r_{j-1}^+) b_i^{(j)}(n,k) + (r_j - d_i) b_i^{(j)}(n,k-1) = (r_j - r_{j-1}^+)\sum_{l\in S} b_l^{(j)}(n-1,k-1)P_{li}$$

... (7)

$$b_i^{(u)}(n,0) = (\acute{\mathbf{a}}P^n)_i,\ \forall i \in S^+$$

... (8)

$$b_i^{(j+1)}(n,0) = b_i^{(j)}(n,n),\ \forall i \notin B_j \text{ and } b_i^{(m)}(n,n) = 0,\ \forall i \notin B_m$$

... (9)

$\boldsymbol{\alpha}$

Since $i \in B_j$ and then $d_i = r_j$, we obtain via Equality (7):

$$\begin{aligned} b_i^{(j)}(n,n) &= \sum_{l\in S} b_l^{(j)}(n-1,n-1)P_{li} \\ b_i^{(j+1)}(n,0) &= \sum_{l\in S} b_l^{(j+1)}(n-1,0)P_{li}. \end{aligned}$$

Keeping into consideration of Identity (9), these two equalities above give:

$$\begin{aligned} b_i^{(j)}(n,n) - b_i^{(j+1)}(n,0) &= \sum_{l\in B_j}\left(b_l^{(j)}(n-1,n-1) - b_l^{(j+1)}(n-1,0)\right)P_{li} \\ &= \left(b_{B_j}^{(j)}(n-1,n-1) - b_{B_j}^{(j+1)}(n-1,0)\right)P_{B_j B_j} e_i . \end{aligned}$$

By induction on the integer n, we get:

$$\begin{aligned} b_i^{(j)}(n,n) - b_i^{(j+1)}(n,0) &= \left(b_{B_j}^{(j)}(0,0) - b_{B_j}^{(j+1)}(0,0)\right)\left(P_{B_j B_j}\right)^n e_i \\ &= \alpha_{B_j} P_{B_j B_j}^n e_i,\ \text{result of Identity (8) and (9).} \end{aligned}$$

By injecting the above equality in the expression of the jump $JF_i(t,x)$, established at the beginning of this proof, we obtain:

$$\begin{aligned} JF_i(t,r_j t) &= \sum_{n\geq 0} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\alpha_{B_j}\left(P_{B_j B_j}\right)^n e_i \\ &= \alpha_{B_j} e^{\lambda t (P_{B_j B_j} - I)} e_i \\ &= \alpha_{B_j} e^{A_{B_j B_j} t} e_i,\ \text{since } P = \frac{A}{\lambda} + I. \end{aligned}$$

The proof of this theorem is then completed.

CONCLUSION

This paper proved the uniqueness of the solution for stochastic fluid models in transient regime under a set of boundary conditions. It is shown that the discontinuity property is crucial to have an exact solution. Unfortunately, this property was ignored in many papers thereby leading to different mathematical errors that we specified in details. Finally, the proof of the theorem that uses the uniformization technique was completed.

ACKNOWLEDGMENT

The authors would to thank the Scientific Research Deanship at King Faisal University for the material and moral support in funding this research project No. 150139.

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تعليقات على الحل الانتقالي لنماذج السوائل العشوائية

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استلام 31 مارس 2015م - قبول 13 نوفمبر 2015م

الملخص

لقد تمت، على نطاق واسع، دراسة التوزيع الاحتمالي المشترك لمستوى السائل في جهاز التخزين وحالة عملية ماركوف المسيطرة على معدلات الإدخال/الإخراج في نموذج السوائل العشوائية. هذا التوزيع الاحتمالي في النظام الانتقالي يناظر نظاما تفاضليا خطيا جزئيا، وقد تم اقتراح العديد من التقنيات والطرق لحل هذا النظام التفاضلي. ولكن وحدانية الحل لم تتم دراستها إطلاقا. في هذه الورقة البحثية، حددت بدقة الشروط الحدية لضمان الوحدانية. تم إثبات أن القفزات بقيم معينة في بعض النقاط الحرجة ضرورية لضمان وحدانية الحل. وجب أيضا التنبيه أن بعض الطرق المستخدمة لتحويل لابلاس لا تأخذ بعين الاعتبار خاصية الانفصالية مما يؤدي إلى أخطاء رياضية مختلفة على حسب النهج المتبع. وأخيرا، تم إثبات أن النهج المستخدم لتقنية الانتظامية (uniformization) يعطي الحل الدقيق الصحيح وتم برهان هذا الحل.

الكلمات المفتاحية: تحويل لابلاس، عملية ماركوف، المعادلات التفاضلية الجزئية، نماذج السوائل العشوائية.