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Determinant of a Neutrosophic Matrix

Eman Alabdullah[1,2] and Safwan Aouira[1]

[1] Department of Mathematics, Faculty of Sciences, Aleppo University, Aleppo, Syria
[2] Faculty of Sciences, Al Furat University, Deir-ez-Zor, the Syrian Arab Republic

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Abstract

In this paper, we study some properties of the determinant of a neutrosophic matrix. Also, we prove that $|A.adj(A)| = |A| = |adj(A).A|$ and define the matrices $A_{(p_1 \dots p_m|q_1 \dots q_m)}$ and $A_{(p \to q)}$. Further, a method is presented for calculating the determinant of a neutrosophic matrix that has a large number of columns and rows.

KEYWORDS

adjoint, identity neutrosophic matrix, permutation, principal submatrix, transpose, triangular NM

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1. Introduction

The Neutrosophic Theorem is a new approach to dealing with issues including imprecise, indeterminant and discordant data.

A neutrosophic set is described philosophically by Smarandache (1998). It is a generalization of the concept of the fuzzy set and the intuitionistic fuzzy set. Each element in the neutrosophic set has three related defining functions that are independent of one another: the membership function (t), indeterminacy function (i) and the non-membership function (f). These three functions are defined on the universe of discourse X (Smarandache, 2006). The definition of the neutrosophic set, known as a single-valued neutrosophic set, was then provided by Wang *et al.* (2010). The single-valued neutrosophic set is applied to algebraic and topological structures. Ç etkin and Aygün (2015) introduced the concepts of neutrosophic subgroups of a given classical group and neutrosophic of a given classical ring (Çetkin and Aygün, 2019). Also, Çetkin *et al.* (2017) presented the definition of the neutrosophic submodule of module and studied some of its fundamental properties.

Molodtsov (1999) established the soft set theory, and it is a new mathematical tool for modelling ambiguity and uncertainty.

Matrices are very crucial to science and technology. However, there are situations when the classical matrix theory is unable to resolve the problems with uncertainties that arise in an uncertain environment.

Dhar *et al.* (2014) introduced a type of neutrosophic matrix, called a square neutrosophic matrix, with entries in the form $a + Ib$ (neutrosophic number) where a, b are the elements in [0,1] and I is an uncertainty such that $I^n = I$; n being a positive integer. Sumathi and Arockiarani (2014) introduced new operations for fuzzy neutrosophic soft matrices. Uma *et al.* (2017) have introduced the determinant and adjoint of square fuzzy neutrosophic soft matrices. A type of matrix termed a neutrosophic matrix, with inputs from a single-valued neutrosophic set, is defined by Varol, *et al.* (2019) along with some algebraic operations describing it. By using the operations component wise addition and component wise multiplication, they have proven that a collection of all neutrosophic matrices forms a semiring.

The determinant of a neutrosophic fuzzy matrix has been introduced by Sophia and Jayapriya (2019), and they have researched its properties. In addition, the trace and the adjoint of a neutrosophic fuzzy matrix are defined. Salama *et al.* (2022) introduced the neutrosophic matrix in a completely different form compared to Dhar *et al.* (2014), where a square neutrosophic matrix of order $n \times n$ is defined as $M = A + BI$, such that A, B are two square real matrices of order $n \times n$.

In this work, we will recall the notion of a single-valued neutrosophic set, which is referred to as a neutrosophic set for convenience. Then we give a brief summary of neutrosophic matrices and several algebraic operations on them. Furthermore, we present a definition of the determinant of a neutrosophic matrix and some of its properties. Finally, we give our conclusions.

2. Preliminaries

In the section, we give some definitions that are used in the paper. First, the operations $\vee$ and $\wedge$ for $a, b \in [0, 1]$ are defined as follows: $a \vee b = max\{a, b\}, a \wedge b = min\{a, b\}$.

2.1. Definition (Wang *et al.*, 2010):

A single-valued neutrosophic set A on the universal set X is defined by the following form: $A = \{\langle x. t_A(x), i_A(x), f_A(x)\rangle : x \in X\}$, where $t_A, i_A, f_A: X \to [0,1]$ define the degree of membership function, the degree of indeterminacy function, and the degree of non-membership function, respectively, for each element $x \in X$ in the set A such that $0 \leq t_A(x) + i_A(x) + f_A(x) \leq 3$.

2.2. Example (Wang *et al.*, 2010):

$X = \{x_1, x_2, x_3\}$, where x_1 is capacity, x_2 is trustworthiness and x_3 is cost. The values of $\{x_1, x_2, x_3\}$ are in [0,1] and are obtained from questionnaires completed by experts. The experts assess their point of view in three combinations: the degree of goodness, the degree of indeterminacy and the degree of poorness to explain the characteristics of the objects. Suppose A is a single-valued neutrosophic set on X , such that $A = \{\langle x_1, 0.3, 0.4, 0.5\rangle, \langle x_2, 0.5, 0.2, 0.3 \rangle, \langle x_3, 0.7, 0.2, 0.2 \rangle\}$, where for x_1, the degree of goodness of capacity is 0.3, the degree of indeterminacy of capacity is 0.4 and the degree of falsity of capacity is 0.5 etc.

Corresponding Author: Eman Alabdullah 00963988113280, emankhder1987@gmail.com

2.3. Definition (Varol *et al.*, 2019):

A neutrosophic matrix of order $m \times n$ is defined by $A = [(t_A(a_{ij}), i_A(a_{ij}), f_A(a_{ij}))]$, such that $t_A(a_{ij}), i_A(a_{ij}), f_A(a_{ij})$ denote truth-membership, indeterminacy-membership and falsity-membership values of the ij-th element in A, satisfying the condition $0 \le t_A(a_{ij}) + i_A(a_{ij}) + f_A(a_{ij}) \le 3$ for all i, j.

2.4. Example:

The following matrix A is a neutrosophic matrix of order 3×1:

$$A = \begin{bmatrix} (0.3, 0.4, 0.5) \\ (0.5, 0.2, 0.3) \\ (0.7, 0.2, 0.2) \end{bmatrix}.$$

2.5. Definition (Varol *et al.*, 2019):

Let $A = [(t_A(a_{ij}), i_A(a_{ij}), f_A(a_{ij}))]$ and

$B = [(t_B(b_{ij}), i_B(b_{ij}), f_B(b_{ij}))]$

be two neutrosophic matrices of order $m \times n$.

Then the matrix addition and subtraction are defined as

$$A + B = [(t_A(a_{ij}) \vee t_B(b_{ij}), i_A(a_{ij}) \vee i_B(b_{ij}), f_A(a_{ij}) \wedge f_B(b_{ij}))].$$

$$A - B = [(t_A(a_{ij}) - t_B(b_{ij}), i_A(a_{ij}) - i_B(b_{ij}), f_A(a_{ij}) - f_B(b_{ij}))],$$

where

$$t_A(a_{ij}) - t_B(b_{ij}) = \begin{cases} t_A(a_{ij}) & ; \ t_A(a_{ij}) \ge t_B(b_{ij}) \\ 0 & ; \ otherwise \end{cases},$$

$$i_A(a_{ij}) - i_B(b_{ij}) = \begin{cases} i_A(a_{ij}) & ; \ i_A(a_{ij}) \ge i_B(b_{ij}) \\ 0 & ; \ otherwise \end{cases},$$

$$f_A(a_{ij}) - f_B(b_{ij}) = \begin{cases} f_A(a_{ij}) & ; \ f_A(a_{ij}) \le f_B(b_{ij}) \\ 1 & ; \ otherwise \end{cases}.$$

And the component wise matrix multiplication is defined by

$$A \bullet B = [(t_A(a_{ij}) \wedge t_B(b_{ij}), i_A(a_{ij}) \wedge i_B(b_{ij}), f_A(a_{ij}) \vee f_B(b_{ij}))].$$

2.6. Definition (Varol *et al.*, 2019):

Let $A = [(t_A(a_{ij}), i_A(a_{ij}), f_A(a_{ij}))]$ and $B = [(t_B(b_{ij}), i_B(b_{ij}), f_B(b_{ij}))]$ be two neutrosophic matrices of order $m \times n$ and $n \times p$, respectively. Then the matrix product AB is defined as

$$AB = \left[\left(\bigvee_{k=1}^{n} t_A(a_{ik}) \wedge t_B(b_{kj}), \bigvee_{k=1}^{n} i_A(a_{ik}) \wedge i_B(b_{kj}), \bigwedge_{k=1}^{n} f_A(a_{ik}) \vee f_B(b_{kj})\right)\right]$$

We can also write,

$$AB = \left[(\sum_{k=1}^{n} t_A(a_{ik}).\, t_B(b_{kj}), \sum_{k=1}^{n} i_A(a_{ik}).\, i_B(b_{kj}), \prod_{k=1}^{n} f_A(a_{ik}) + f_B(b_{kj}))\right]$$

In this case, A and B are conformable for multiplication.

2.7. Definition (Varol *et al.*, 2019):

Let $A = [(t_A(a_{ij}), i_A(a_{ij}), f_A(a_{ij}))]$ be a neutrosophic matrix of order $m \times n$. Then the transpose of A is defined by

$$A^T = [(t_A(a_{ji}), i_A(a_{ji}), f_A(a_{ji}))].$$

2.8. Definition (Varol *et al.*, 2019):

Let $A = [(t_A(a_{ij}), i_A(a_{ij}), f_A(a_{ij}))]$ be a neutrosophic matrix of order $m \times n$ and $k \in [0,1]$. Then the neutrosophic scalar multiplication is defined as

$$kA = [(k \wedge t_A(a_{ij}), k \wedge i_A(a_{ij}), (1-k) \vee f_A(a_{ij}))].$$

2.9. Definition (Varol *et al.*, 2019):

Let A be a neutrosophic matrix of order $m \times n$.

If all its entries are $(0,0,1)$, then A is said to be zero neutrosophic matrix and denoted by $\boldsymbol{O}$.

If all its entries are $(1,1,0)$, then A is said to be universal neutrosophic matrix and denoted by $\boldsymbol{J}$.

2.10. Definition (Varol *et al.*, 2019):

The identity neutrosophic matrix of order $n \times n$ is denoted by $\boldsymbol{I}_n$, and defined as $\boldsymbol{I}_n = [(t_{I_n}(\lambda_{ij}), i_{I_n}(\lambda_{ij}), f_{I_n}(\lambda_{ij}))]$,

where $\left(t_{I_n}(\lambda_{ij}), i_{I_n}(\lambda_{ij}), f_{I_n}(\lambda_{ij})\right) = \begin{cases} (0,0,1) \ ; \ i \ne j \\ (1,1,0) \ ; \ i = j \end{cases}$.

2.11. Definition (Sophia and Jayapriya, 2019):

Let A be a neutrosophic matrix of order $n \times n$.

- If $\left(t_A(a_{ij}), i_A(a_{ij}), f_A(a_{ij})\right) = (0,0,1) \ \forall i > j$, then the matrix A is called an upper triangular neutrosophic matrix.
- If $\left(t_A(a_{ij}), i_A(a_{ij}), f_A(a_{ij})\right) = (0,0,1) \ \forall i < j$, then the matrix A is called a lower triangular neutrosophic matrix.
- The matrix A is called a triangular neutrosophic matrix if either $\left(t_A(a_{ij}), i_A(a_{ij}), f_A(a_{ij})\right) = (0,0,1) \ \forall i > j$ or $\left(t_A(a_{ij}), i_A(a_{ij}), f_A(a_{ij})\right) = (0,0,1) \ \forall i < j$.

3. Determinant of a Neutrosophic Matrix

3.1. Definition (Sophia and Jayapriya, 2019):

The determinant of a neutrosophic matrix A of order $n \times n$ is denoted by det (A) or $|A|$ and is defined by

$$|A| = \sum_{\sigma \in S_n} \left(t_A(a_{1\sigma(1)}), i_A(a_{1\sigma(1)}), f_A(a_{1\sigma(1)})\right). \ldots .\left(t_A(a_{n\sigma(n)}), i_A(a_{n\sigma(n)}), f_A(a_{n\sigma(n)})\right)$$

We can also write,

$$|A| = \begin{pmatrix} \bigvee_{\sigma \in S_n} t_A(a_{1\sigma(1)}) \wedge \ldots \wedge t_A(a_{n\sigma(n)}), \bigvee_{\sigma \in S_n} i_A(a_{1\sigma(1)}) \wedge \ldots \wedge i_A(a_{n\sigma(n)}), \\ \bigwedge_{\sigma \in S_n} f_A(a_{1\sigma(1)}) \vee \ldots \vee f_A(a_{n\sigma(n)}) \end{pmatrix}$$

where S_n is the symmetric group of all permutations of $\{1,2,\ldots,n\}$.

3.2. Example:

Let A be a neutrosophic matrix of order 2×2 such that

$A = \begin{bmatrix} (0.3,0.1,0.2) & (0.1,0.4,0.5) \\ (0,0.1,0.2) & (0.9,0.1,0.1) \end{bmatrix}$. Then

$$\begin{aligned} |A| &= (0.3,0.1,0.2).(0.9,0.1,0.1) + (0.1,0.4,0.5).(0,0.1,0.2) \\ &= (0.3, 0.1, 0.2) + (0, 0.1, 0.5) \\ &= (0.3, 0.1, 0.2). \end{aligned}$$

3.3. Properties of the Determinant of a Neutrosophic Matrix:

<u>3.3.1. Property 1 (Property of Reflection)</u>

The value of the determinant of a neutrosophic matrix remains unchanged if any two rows (columns) are swapped.

<u>3.3.2. Property 2 (Property of All Zero)</u>

If there is a row (column) in a neutrosophic matrix A with all its elements as $(0,0,1)$, then $|A| = (0,0,1)$.

3.3.3. Property 3 (Property of Scalar Multiple)

If there is a row (column) in a neutrosophic matrix A with all its elements multiplied by a non-zero constant, then the determinant gets multiplied by the same constant.

3.3.4. Property 4 (Property of Triangle)

Let A be a triangular neutrosophic matrix of order $n \times n$. Then

$$|A| = \prod_{i=1}^{n} \left(t_A(a_{ii}), i_A(a_{ii}), f_A(a_{ii}) \right).$$

3.3.5. Property 5 (Property of Transpose)

Let A be a neutrosophic matrix of order $n \times n$. Then $|A| = |A^T|$.

3.4. Definition (Sophia and Jayapriya, 2019):

Let A be a neutrosophic matrix of order $n \times n$ and A_{ij} is the neutrosophic matrix of order $(n-1) \times (n-1)$ formed by deleting row i and column j from A. The adjoint matrix of A is denoted by $adj(A)$ and is defined by $adj(A) = [|A_{ji}|]$.

4. Main Results

4.1. Theorem:

Let $A = \left[\left(t_A(a_{ij}), i_A(a_{ij}), f_A(a_{ij})\right)\right]$ be a neutrosophic matrix of order $n \times n$. If $(t_A(a_{ii}), i_A(a_{ii}), f_A(a_{ii})) \geq (t_A(a_{ik}), i_A(a_{ik}), f_A(a_{ik}))$; $k = 1,2,\ldots,n$ for all $1 \leq i \leq n$, then $|A| = (t_A(a_{11}), i_A(a_{11}), f_A(a_{11})) \ldots (t_A(a_{nn}), i_A(a_{nn}), f_A(a_{nn}))$.

Proof

By definition of $|A|$, we get

$$|A| \geq \left(t_A(a_{11}), i_A(a_{11}), f_A(a_{11})\right) \ldots \left(t_A(a_{nn}), i_A(a_{nn}), f_A(a_{nn})\right) \quad (1)$$

For any permutation $\sigma \in S_n$, we have

$$\left(t_A(a_{ii}), i_A(a_{ii}), f_A(a_{ii})\right) \geq \left(t_A(a_{i\sigma(i)}), i_A(a_{i\sigma(i)}), f_A(a_{i\sigma(i)})\right); \quad i = 1,2,\ldots,n$$

Since $\left(t_A(a_{ii}), i_A(a_{ii}), f_A(a_{ii})\right) \geq \left(t_A(a_{ik}), i_A(a_{ik}), f_A(a_{ik})\right)$;

$k = 1,2,\ldots,n$ for all $1 \leq i \leq n$.

Hence

$$\left(t_A(a_{11}), i_A(a_{11}), f_A(a_{11})\right) \ldots \left(t_A(a_{nn}), i_A(a_{nn}), f_A(a_{nn})\right) \geq \left(t_A(a_{1\sigma(1)}), i_A(a_{1\sigma(1)}), f_A(a_{1\sigma(1)})\right). \ldots . \left(t_A(a_{n\sigma(n)}), i_A(a_{n\sigma(n)}), f_A(a_{n\sigma(n)})\right)$$

$$\Rightarrow \left(t_A(a_{11}), i_A(a_{11}), f_A(a_{11})\right) \ldots \left(t_A(a_{nn}), i_A(a_{nn}), f_A(a_{nn})\right) \geq \sum_{\sigma \in S_n} \left(t_A(a_{1\sigma(1)}), i_A(a_{1\sigma(1)}), f_A(a_{1\sigma(1)})\right). \ldots . \left(t_A(a_{n\sigma(n)}), i_A(a_{n\sigma(n)}), f_A(a_{n\sigma(n)})\right) = |A| \quad (2)$$

$$\Rightarrow |A| = \left(t_A(a_{11}), i_A(a_{11}), f_A(a_{11})\right) \ldots \left(t_A(a_{nn}), i_A(a_{nn}), f_A(a_{nn})\right);$$

by (1) & (2).

4.2. Theorem:

Let $A = \left[\left(t_A(a_{ij}), i_A(a_{ij}), f_A(a_{ij})\right)\right]$ be a neutrosophic matrix of order $n \times n$. Then $|AA^T| \geq |A|$.

Proof

Let $AA^T = [(t_{AA^T}(p_{ij}), i_{AA^T}(p_{ij}), f_{AA^T}(p_{ij}))]$, where

$$\left(t_{AA^T}(p_{ij}), i_{AA^T}(p_{ij}), f_{AA^T}(p_{ij})\right) = \left(\sum_{k=1}^{n} t_A(a_{ik}).t_{A^T}(a_{kj}), \sum_{k=1}^{n} i_A(a_{ik}).i_{A^T}(a_{kj}), \prod_{k=1}^{n} f_A(a_{ik}) + f_{A^T}(a_{kj})\right)$$

$$= \left(\sum_{k=1}^{n} t_A(a_{ik}).t_A(a_{jk}), \sum_{k=1}^{n} i_A(a_{ik}).i_A(a_{jk}), \prod_{k=1}^{n} f_A(a_{ik}) + f_A(a_{jk})\right)$$

for all $i, j \in \{1,2,\ldots,n\}$.

If $i = j$, we see that

$$\left(t_{AA^T}(p_{ii}), i_{AA^T}(p_{ii}), f_{AA^T}(p_{ii})\right) = \left(\sum_{k=1}^{n} t_A(a_{ik}), \sum_{k=1}^{n} i_A(a_{ik}), \prod_{k=1}^{n} f_A(a_{ik})\right) = \sum_{k=1}^{n} \left(t_A(a_{ik}), i_A(a_{ik}), f_A(a_{ik})\right)$$

For any permutation $\sigma \in S_n$, we get

$$\sum_{k=1}^{n} \left(t_A(a_{ik}), i_A(a_{ik}), f_A(a_{ik})\right) \geq \left(t_A(a_{i\sigma(i)}), i_A(a_{i\sigma(i)}), f_A(a_{i\sigma(i)})\right); \quad i \in \{1,2,\ldots,n\}$$

$$|AA^T| = \sum_{\sigma \in S_n} \left(t_{AA^T}(p_{1\sigma(1)}), i_{AA^T}(p_{1\sigma(1)}), f_{AA^T}(p_{1\sigma(1)})\right). \ldots . \left(t_{AA^T}(p_{n\sigma(n)}), i_{AA^T}(p_{n\sigma(n)}), f_{AA^T}(p_{n\sigma(n)})\right) \geq$$

$$\left(t_{AA^T}(p_{11}), i_{AA^T}(p_{11}), f_{AA^T}(p_{11})\right) \ldots \left(t_{AA^T}(p_{nn}), i_{AA^T}(p_{nn}), f_{AA^T}(p_{nn})\right)$$

$$= \left(\sum_{k=1}^{n} \left(t_A(a_{1k}), i_A(a_{1k}), f_A(a_{1k})\right)\right) \ldots . \left(\sum_{k=1}^{n} \left(t_A(a_{nk}), i_A(a_{nk}), f_A(a_{nk})\right)\right)$$

$$\geq \left(t_A(a_{1\sigma(1)}), i_A(a_{1\sigma(1)}), f_A(a_{1\sigma(1)})\right). \ldots . \left(t_A(a_{n\sigma(n)}), i_A(a_{n\sigma(n)}), f_A(a_{n\sigma(n)})\right)$$

Thus,

$$|AA^T| \geq \sum_{\sigma \in S_n} \left(t_A(a_{1\sigma(1)}), i_A(a_{1\sigma(1)}), f_A(a_{1\sigma(1)})\right). \ldots . \left(t_A(a_{n\sigma(n)}), i_A(a_{n\sigma(n)}), f_A(a_{n\sigma(n)})\right) = |A|.$$

4.3. Remark:

In general, we have that $|AB| \neq |A||B|$ where A and B are two neutrosophic matrices of order $n \times n$. This is illustrated in the next example.

4.4. Example:

Let $A = \begin{bmatrix} (0.14,0.7,0.1) & (0.25,0.6,0.12) \\ (0.12,0.7,0.3) & (0.24,0.7,0.1) \end{bmatrix}$ and

$B = \begin{bmatrix} (0.5,0.3,0) & (0.3,0.5,0.2) \\ (0.2,0.6,0) & (0.16,0.7,0.4) \end{bmatrix}$.

Then $AB = \begin{bmatrix} (0.2,0.6,0.1) & (0.16,0.6,0.2) \\ (0.2,0.6,0.1) & (0.16,0.7,0.3) \end{bmatrix}$.

Therefore, $|A| = (0.14,0.7,0.1)$, $|B| = (0.2,0.5,0.2)$,

$|A|.|B| = (0.14,0.5,0.2)$ and $|A.B| = (0.16,0.6,0.2)$.

We notice that $|AB| \neq |A||B|$.

4.5. Theorem:

Let A and B be two neutrosophic matrices of order $n\times n$. Then

1) $|AB| \geq |A||B|$,
2) $|AB| \geq |A+B|$.

Proof

1) $|AB| =$

$$\left[(\sum_{k=1}^{n} t_A(a_{ik}).t_B(b_{kj}), \sum_{k=1}^{n} i_A(a_{ik}).i_B(b_{kj}), \prod_{k=1}^{n} f_A(a_{ik})+f_B(b_{kj}))\right]$$

we have

$$|AB| = \sum_{\sigma\in S_n}\left(\sum_{k=1}^{n} t_A(a_{1k}).t_B(b_{k\sigma(1)}), \sum_{k=1}^{n} i_A(a_{1k}).i_B(b_{k\sigma(1)}), \prod_{k=1}^{n} f_A(a_{1k}) + f_B(b_{k\sigma(1)})\right).\dots$$
$$.\left(\sum_{k=1}^{n} t_A(a_{nk}).t_B(b_{k\sigma(n)}), \sum_{k=1}^{n} i_A(a_{nk}).i_B(b_{k\sigma(n)}), \prod_{k=1}^{n} f_A(a_{nk}) + f_B(b_{k\sigma(n)})\right)$$

$$= \sum_{\sigma\in S_n}\left(\sum_{k_1,\dots,k_n} t_A(a_{1k_1})\dots t_A(a_{nk_n}).t_B(b_{k_1\sigma(1)})\dots t_B(b_{k_n\sigma(n)}), \sum_{k_1,\dots,k_n} i_A(a_{1k_1})\dots i_A(a_{nk_n}).i_B(b_{k_1\sigma(1)})\dots i_B(b_{k_n\sigma(n)}), \prod_{k_1,\dots,k_n} f_A(a_{1k_1})\dots f_A(a_{nk_n}) + f_B(b_{k_1\sigma(1)})\dots f_B(b_{k_n\sigma(n)})\right)$$

$$= \sum_{\sigma\in S_n}\sum_{k_1,\dots,k_n}\left(t_A(a_{1k_1}), i_A(a_{1k_1}), f_A(a_{1k_1})\right).\dots \left(t_A(a_{nk_n}), i_A(a_{nk_n}), f_A(a_{nk_n})\right).\left(t_B(b_{k_1\sigma(1)}), i_B(b_{k_1\sigma(1)}), f_B(b_{k_1\sigma(1)})\right). \dots.\left(t_B(b_{k_n\sigma(n)}), i_B(b_{n\sigma(n)}), f_B(b_{k_n\sigma(n)})\right)$$

$$= \sum_{k_1,\dots,k_n}\left(t_A(a_{1k_1}), i_A(a_{1k_1}), f_A(a_{1k_1})\right)\dots\left(t_A(a_{nk_n}), i_A(a_{nk_n}), f_A(a_{nk_n})\right).\sum_{\sigma\in S_n}\left(t_B(b_{k_1\sigma(1)}), i_B(b_{k_1\sigma(1)}), f_B(b_{k_1\sigma(1)})\right).\dots .\left(t_B(b_{k_n\sigma(n)}), i_B(b_{k_n\sigma(n)}), f_B(b_{k_n\sigma(n)})\right)$$

$$\geq \sum_{k_1,\dots,k_n}\left(t_A(a_{1k_1}), i_A(a_{1k_1}), f_A(a_{1k_1})\right)\dots\left(t_A(a_{nk_n}), i_A(a_{nk_n}), f_A(a_{nk_n})\right).\sum_{\sigma\in S_n}\left(t_B(b_{1\sigma(1)}), i_B(b_{1\sigma(1)}), f_B(b_{1\sigma(1)})\right).\dots .\left(t_B(b_{n\sigma(n)}), i_B(b_{n\sigma(n)}), f_B(b_{n\sigma(n)})\right)$$

$$= \sum_{(k_1,\dots,k_n)\in S_n}\left(t_A(a_{1k_1}), i_A(a_{1k_1}), f_A(a_{1k_1})\right)\dots\left(t_A(a_{nk_n}), i_A(a_{nk_n}), f_A(a_{nk_n})\right).|B|$$

$$= |A||B|.$$

2) $|AB| =$

$$\sum_{\sigma\in S_n}\left(\sum_{k=1}^{n} t_A(a_{1k}).t_B(b_{k\sigma(1)}), \sum_{k=1}^{n} i_A(a_{1k}).i_B(b_{k\sigma(1)}), \prod_{k=1}^{n} f_A(a_{1k}) + f_B(b_{k\sigma(1)})\right).\dots$$
$$.\left(\sum_{k=1}^{n} t_A(a_{nk}).t_B(b_{k\sigma(n)}), \sum_{k=1}^{n} i_A(a_{nk}).i_B(b_{k\sigma(n)}), \prod_{k=1}^{n} f_A(a_{nk}) + f_B(b_{k\sigma(n)})\right)$$

$$= \sum_{\sigma\in S_n}\left(\bigwedge_{1\leq s,t\leq n}\left(t_A(a_{1s})\vee t_B(b_{t\sigma(1)})\right), \bigwedge_{1\leq s,t\leq n}\left(i_A(a_{1s})\vee i_B(b_{t\sigma(n)})\right), \bigvee_{1\leq s,t\leq n}\left(f_A(a_{1s})\wedge f_B(b_{t\sigma(1)})\right)\right)\dots\left(\bigwedge_{1\leq s,t\leq n}\left(t_A(a_{ns})\vee t_B(b_{t\sigma(n)})\right), \bigwedge_{1\leq s,t\leq n}\left(i_A(a_{ns})\vee i_B(b_{t\sigma(n)})\right), \bigvee_{1\leq s,t\leq n}\left(f_A(a_{ns})\wedge f_B(b_{t\sigma(n)})\right)\right)$$

$$\leq \sum_{\sigma\in S_n}\left(t_A(a_{1\sigma(1)})\vee t_B(b_{1\sigma(1)}), i_A(a_{1\sigma(1)})\vee i_B(b_{1\sigma(1)}), f_A(a_{1\sigma(1)})\wedge f_B(b_{1\sigma(1)})\right)\dots\left(t_A(a_{n\sigma(n)})\vee t_B(b_{n\sigma(n)}), i_A(a_{n\sigma(n)})\vee i_B(b_{n\sigma(n)}), f_A(a_{n\sigma(n)})\wedge f_B(b_{n\sigma(n)})\right)$$

$$= |A+B|.$$

4.6. Corollary:

(1) Let $A_1,\dots,A_m$ be neutrosophic matrices of order $n\times n$. Then

$$|A_1|.\dots.|A_m| \leq |A_1.\dots.A_m| \leq \left|\sum_{k=1}^{m} A_k\right| ;\ m\in\mathbb{N}.$$

(2) Let A be a neutrosophic matrix of order $n\times n$. Then

$$|A^r| = |A| ;\ r\in\mathbb{N}.$$

4.7. Theorem:

Let A, B *and* C be three neutrosophic matrices of order $n\times n$. Then $\begin{vmatrix} A & C \\ \boldsymbol{O} & B\end{vmatrix} = |A|.|B|$, where $\boldsymbol{O} = [(0,0,1)]$ is the zero neutrosophic matrix of order $n\times n$.

Proof

Suppose that $\begin{bmatrix} A & C \\ \boldsymbol{O} & B\end{bmatrix} = D = \left[\left(t_D(d_{ij}), i_D(d_{ij}), f_D(d_{ij})\right)\right]$, then,

$$\begin{vmatrix} A & C \\ \boldsymbol{O} & B\end{vmatrix} = \sum_{\sigma\in S_{2n}}\left(t_D(d_{1\sigma(1)}), i_D(d_{1\sigma(1)}), f_D(d_{1\sigma(1)})\right).\dots.\left(t_D(d_{2n\sigma(2n)}), i_D(d_{2n\sigma(2n)}), f_D(d_{2n\sigma(2n)})\right)$$

$$= \sum_{\substack{\sigma\in S_{2n}\\ \sigma(i)\leq n\ ;\ i\leq n}}\left(t_D(d_{1\sigma(1)}), i_D(d_{1\sigma(1)}), f_D(d_{1\sigma(1)})\right)\dots.\left(t_D(d_{2n\sigma(2n)}), i_D(d_{2n\sigma(2n)}), f_D(d_{2n\sigma(2n)})\right) + \sum_{\substack{\sigma\in S_{2n}\\ \exists i>n\ ;\ \sigma(i)\leq n}}\left(t_D(d_{1\sigma(1)}), i_D(d_{1\sigma(1)}), f_D(d_{1\sigma(1)})\right)\dots.\left(t_D(d_{2n\sigma(2n)}), i_D(d_{2n\sigma(2n)}), f_D(d_{2n\sigma(2n)})\right)$$

Since, for any permutation $\sigma\in S_{2n}$ such that $\exists i>n\,;\sigma(i)\leq n$, there is that

$$\left(t_D(d_{i\sigma(i)}), i_D(d_{i\sigma(i)}), f_D(d_{i\sigma(i)})\right) = (0,0,1)$$

Therefore,

$$\begin{vmatrix} A & C \\ O & B \end{vmatrix} = \sum_{\substack{\sigma\in S_{2n} \\ \sigma(i)\le n\,;\, i\le n}} \left(t_D(d_{1\sigma(1)}), i_D(d_{1\sigma(1)}), f_D(d_{1\sigma(1)})\right) \dots . \left(t_D(d_{2n\sigma(2n)}), i_D(d_{2n\sigma(2n)}), f_D(d_{2n\sigma(2n)})\right) =$$

$$\sum_{\substack{\alpha\in S_n \\ \alpha(i)=\sigma(i) \\ ;\, i\le n}} \left(t_D(d_{1\alpha(1)}), i_D(d_{1\alpha(1)}), f_D(d_{1\alpha(1)})\right) \dots \left(t_D(d_{n\alpha(n)}), i_D(d_{n\alpha(n)}), f_D(d_{n\alpha(n)})\right) . \sum_{\substack{\sigma\in S_{2n} \\ \beta(i)=\sigma(i+n) \\ ;\, i\le n}} \left(t_D(d_{1\beta(1)}), i_D(d_{1\beta(1)}), f_D(d_{1\beta(1)})\right) . \dots \left(t_D(d_{n\beta(n)}), i_D(d_{n\beta(n)}), f_D(d_{n\beta(n)})\right)$$

$= |A|.|B|$.

4.8. Theorem:

Let A and B be two neutrosophic matrices of order $n\times n$. If both A and B are upper triangular neutrosophic matrices or both lower triangular neutrosophic matrices, then $|AB| = |A||B|$.

Proof

Let A and B be two upper triangular neutrosophic matrices of order $n\times n$.

Assume that $AB = \left[\left(t_{AB}(d_{ij}), i_{AB}(d_{ij}), f_{AB}(d_{ij})\right)\right]$. The ij-th element of the product AB is $\left(t_{AB}(d_{ij}), i_{AB}(d_{ij}), f_{AB}(d_{ij})\right) =$

$$\left(\sum_{k=1}^{n} t_A(a_{ik}).t_B(b_{kj}), \sum_{k=1}^{n} i_A(a_{ik}).i_B(b_{kj}), \prod_{k=1}^{n} f_A(a_{ik}) + f_B(b_{kj})\right)\ ; i,j = 1,2,\dots,n$$

For $i > j$: if $k > i$ then $k > j$ so that $\left(t_B(b_{kj}), i_B(b_{kj}), f_B(b_{kj})\right) = (0,0,1)$ and if $i > k$ then $\left(t_A(a_{ik}), i_A(a_{ik}), i_A(a_{ik})\right) = (0,0,1)$, hence

$$t_{AB}(d_{ij}) = \sum_{k=1}^{n} t_A(a_{ik}).t_B(b_{kj}) = \sum_{\substack{k=1 \\ k>i}}^{n} t_A(a_{ik}).(0) + \sum_{\substack{k=1 \\ k<i}}^{n} (0).t_B(b_{kj}) = 0$$

Similarly, we obtain

$i_{AB}(d_{ij}) = 0$ & $f_{AB}(d_{ij}) = 1$

Thus,

$$\left(t_{AB}(d_{ij}), i_{AB}(d_{ij}), f_{AB}(d_{ij})\right) = (0,0,1)\ \ \forall i > j$$

This means that AB is an upper triangular neutrosophic matrix.

Therefore,

$$|AB| = \prod_{i=1}^{n} \left(t_{AB}(d_{ii}), i_{AB}(d_{ii}), f_{AB}(d_{ii})\right)$$

Now, we have

$$t_A(a_{ik}).t_B(b_{ki}) = \begin{cases} (0).t_B(b_{ki})\ ;\ i > k \\ t_A(a_{ik}).(0)\ ;\ i < k \\ t_A(a_{ii}).t_B(b_{ii})\ ;\ i = k \end{cases}$$

$$\Rightarrow\ t_A(a_{ik}).t_B(b_{ki}) = \begin{cases} 0 & ;\ i \ne k \\ t_A(a_{ii}).t_B(b_{ii}) & ;\ i = k \end{cases}$$

$$\Rightarrow\ t_{AB}(d_{ii}) = \sum_{k=1}^{n} t_A(a_{ik}).t_B(b_{ki}) = t_A(a_{ii}).t_B(b_{ii})$$

Similarly, we obtain,

$i_{AB}(d_{ii}) = i_A(a_{ii}).i_B(b_{ii})$ & $f_{AB}(d_{ii}) = f_A(a_{ii}) + f_B(b_{ii})$

Thus,

$$|AB| = \prod_{i=1}^{n} \left(t_A(a_{ii}).t_B(b_{ii}), i_A(a_{ii}).i_B(b_{ii}), f_A(a_{ii}) + f_B(b_{ii})\right)$$

$$= \prod_{i=1}^{n} \left(t_A(a_{ii}), i_A(a_{ii}), f_A(a_{ii})\right) . \left(t_B(b_{ii}), i_B(b_{ii}), f_B(b_{ii})\right)$$

$$= \left(\prod_{i=1}^{n} \left(t_A(a_{ii}), i_A(a_{ii}), f_A(a_{ii})\right)\right) . \left(\prod_{i=1}^{n} \left(t_B(b_{ii}), i_B(b_{ii}), f_B(b_{ii})\right)\right)$$

$= |A|.|B|$

Similarly, we can prove this property for lower triangular neutrosophic matrices of order $n\times n$.

4.9. Remark:

Let A and B be two neutrosophic matrices of order $n\times n$. If both A and B are upper (lower) triangular neutrosophic matrices, then AB is an upper (lower) triangular neutrosophic matrix.

4.10. Theorem:

Let A be a neutrosophic matrix of order $n\times n$. Then

$$|A| = \sum_{t=1}^{n} \left(t_A(a_{it}), i_A(a_{it}), i_A(a_{it})\right) . |A_{it}|\ ;\ \ i = 1,2,\dots,n$$

Proof

$$|A| = \sum_{\sigma\in S_n} \left(t_A(a_{1\sigma(1)}), i_A(a_{1\sigma(1)}), f_A(a_{1\sigma(1)})\right) . \dots . \left(t_A(a_{n\sigma(n)}), i_A(a_{n\sigma(n)}), f_A(a_{n\sigma(n)})\right)$$

$$= \sum_{t=1}^{n} \sum_{\substack{\sigma\in S_n \\ \sigma(i)=t}} \left(t_A(a_{1\sigma(1)}), i_A(a_{1\sigma(1)}), f_A(a_{1\sigma(1)})\right) . \ \dots . \left(t_A(a_{n\sigma(n)}), i_A(a_{n\sigma(n)}), f_A(a_{n\sigma(n)})\right) =$$

$$\sum_{t=1}^{n} \left(t_A(a_{it}), i_A(a_{it}), f_A(a_{it})\right) . \sum_{\beta\in S_{n_i n_t}} \left(t_A(a_{1\beta(1)}), i_A(a_{1\beta(1)}), f_A(a_{1\beta(1)})\right) \dots \left(t_A(a_{i-1\,\beta(i-1)}), i_A(a_{i-1\,\beta(i-1)}), f_A(a_{i-1\,\beta(i-1)})\right) . \left(t_A(a_{i+1\,\beta(i+1)}), i_A(a_{i+1\,\beta(i+1)}), f_A(a_{i+1\,\beta(i+1)})\right) \dots \left(t_A(a_{n\beta(n)}), i_A(a_{n\beta(n)}), f_A(a_{n\beta(n)})\right)$$

Where $n_i = \{1,2,\dots,n\}\backslash\{i\}$ and $S_{n_i n_t}$ denotes the set of all permutations of the set n_i on the set n_t.

By definition of the determinant, we write

$$|A_{it}| = \sum_{\beta\in S_{n_i n_t}} \left(t_A(a_{1\beta(1)}), i_A(a_{1\beta(1)}), f_A(a_{1\beta(1)})\right) . \dots . \left(t_A(a_{i-1\,\beta(i-1)}), i_A(a_{i-1\,\beta(i-1)}), f_A(a_{i-1\,\beta(i-1)})\right) . \left(t_A(a_{i+1\,\beta(i+1)}), i_A(a_{i+1\,\beta(i+1)}), f_A(a_{i+1\,\beta(i+1)})\right) \dots \left(t_A(a_{n\beta(n)}), i_A(a_{n\beta(n)}), f_A(a_{n\beta(n)})\right)$$

The Proof is complete.

4.11. Definition:

Let A be a neutrosophic matrix of order $n \times n$. The matrix $A_{\binom{p_1 \dots p_m}{q_1 \dots q_m}}$ is the neutrosophic matrix of order $(n-m) \times (n-m)$ that is a result of A by deleting the row p_1,..., the row p_m, the column q_1,... and the column q_m from A, where $p_1 < \dots < p_m$ and $q_1 < \dots < q_m$.

The matrix $A_{\binom{p_1 \dots p_m}{p_1 \dots p_m}}$ called a principal submatrix of order $(n-m) \times (n-m)$ of A.

4.12. Theorem:

Let A be a neutrosophic matrix of order $n \times n$. Then

$$|A| = \sum_{p<q} \left|\begin{matrix} \left(t_A(a_{1p}), i_A(a_{1p}), f_A(a_{1p})\right) & \left(t_A(a_{1q}), i_A(a_{1q}), f_A(a_{1q})\right) \\ \left(t_A(a_{2p}), i_A(a_{2p}), f_A(a_{2p})\right) & \left(t_A(a_{2q}), i_A(a_{2q}), f_A(a_{2q})\right) \end{matrix}\right| . \left|A_{\binom{1\ 2}{p\ q}}\right|$$

where the summation is taken over all p and q in$\{1,2, \dots, n\}$ such that $p < q$.

Proof

$$|A| = \sum_{\sigma \in S_n} \left(t_A(a_{1\sigma(1)}), i_A(a_{1\sigma(1)}), f_A(a_{1\sigma(1)})\right). \dots$$

$$. \left(t_A(a_{n\sigma(n)}), i_A(a_{n\sigma(n)}), f_A(a_{n\sigma(n)})\right)$$

$$= \sum_{\substack{\sigma \in S_n \\ \sigma\{1,2\}=\{1,q\}_{q>1}}} \left(t_A(a_{1\sigma(1)}), i_A(a_{1\sigma(1)}), f_A(a_{1\sigma(1)})\right). \dots$$

$$. \left(t_A(a_{n\sigma(n)}), i_A(a_{n\sigma(n)}), f_A(a_{n\sigma(n)})\right)$$

$$+ \sum_{\substack{\sigma \in S_n \\ \sigma\{1,2\}=\{2,q\}_{q>2}}} \left(t_A(a_{1\sigma(1)}), i_A(a_{1\sigma(1)}), f_A(a_{1\sigma(1)})\right). \dots$$

$$. \left(t_A(a_{n\sigma(n)}), i_A(a_{n\sigma(n)}), f_A(a_{n\sigma(n)})\right)$$

$$+ \cdots + \sum_{\substack{\sigma \in S_n \\ \sigma\{1,2\}=\{n-1,q\}_{q>n-1}}} \left(t_A(a_{1\sigma(1)}), i_A(a_{1\sigma(1)}), f_A(a_{1\sigma(1)})\right). \dots$$

$$. \left(t_A(a_{n\sigma(n)}), i_A(a_{n\sigma(n)}), f_A(a_{n\sigma(n)})\right)$$

$$= \sum_{p<q} \sum_{\substack{\sigma \in S_n \\ \sigma\{1,2\}=\{p,q\}}} \left(t_A(a_{1\sigma(1)}), i_A(a_{1\sigma(1)}), f_A(a_{1\sigma(1)})\right). \dots$$

$$. \left(t_A(a_{n\sigma(n)}), i_A(a_{n\sigma(n)}), f_A(a_{n\sigma(n)})\right)$$

Let $S(p,q) = \{\sigma: \{1,2\} \to \{p,q\} | \ \sigma \ is \ bijection\}$. Then

$$|A| = \sum_{p<q} \left(\sum_{\beta \in S(p,q)} \left(t_A(a_{1\beta(1)}), i_A(a_{1\beta(1)}), f_A(a_{1\beta(1)})\right) \left(t_A(a_{2\beta(2)}), i_A(a_{2\beta(2)}) \right.\right.$$

$$\left.\left. , f_A(a_{2\beta(2)})\right) . \sum_{\substack{\sigma \in S_n \\ \sigma(i) \notin \{p,q\}}} \prod_{i=3}^{n} \left(t_A(a_{i\sigma(i)}), i_A(a_{i\sigma(i)}), f_A(a_{i\sigma(i)})\right) \right)$$

$$= \sum_{p<q} \left|\begin{matrix} \left(t_A(a_{1p}), i_A(a_{1p}), f_A(a_{1p})\right) & \left(t_A(a_{1q}), i_A(a_{1q}), f_A(a_{1q})\right) \\ \left(t_A(a_{2p}), i_A(a_{2p}), f_A(a_{2p})\right) & \left(t_A(a_{2q}), i_A(a_{2q}), f_A(a_{2q})\right) \end{matrix}\right| . \left|A_{\binom{1\ 2}{p\ q}}\right|$$

4.13. Theorem:

Let A be a neutrosophic matrix of order $n \times n$. Then

$$|A| = \sum_{p_1 < p_2 < \dots < p_k} \left|\begin{matrix} \left(t_A(a_{1p_1}), i_A(a_{1p_1}), f_A(a_{1p_1})\right) & \dots & \left(t_A(a_{1p_k}), i_A(a_{1p_k}), f_A(a_{1p_k})\right) \\ \vdots & \ddots & \vdots \\ \left(t_A(a_{kp_1}), i_A(a_{kp_1}), f_A(a_{kp_1})\right) & & \left(t_A(a_{kp_k}), i_A(a_{kp_k}), f_A(a_{kp_k})\right) \end{matrix}\right| . \left|A_{\binom{1\ \ 2\ \dots\ k}{p_1\ p_2 \dots p_k}}\right|$$

where the summation is taken over all $p_1, p_2, \dots, p_k \in \{1,2, \dots, n\}$ such that $p_1 < p_2 < \cdots < p_k$.

$A_{\binom{1\,2\dots k}{p_1\ p_2 \dots p_k}}$ is the matrix obtained from A by striking out the row1, the row 2, ..., the row k, the column p_1, ..., and the column p_k.

Proof

The Proof of this theorem is like the Proof of theorem 4.12.

4.14. Lemma:

Let $A = \begin{bmatrix} \left(t_A(a_{11}), i_A(a_{11}), f_A(a_{11})\right) & \left(t_A(a_{12}), i_A(a_{12}), f_A(a_{12})\right) \\ \left(t_A(a_{21}), i_A(a_{21}), f_A(a_{21})\right) & \left(t_A(a_{22}), i_A(a_{22}), f_A(a_{22})\right) \end{bmatrix}$ be a neutrosophic matrix. Then

$$\left|\begin{matrix} \left(t_A(a_{11}), i_A(a_{11}), f_A(a_{11})\right) & \left(t_A(a_{12}), i_A(a_{12}), f_A(a_{12})\right) \\ \left(t_A(a_{11}), i_A(a_{11}), f_A(a_{21})\right) & \left(t_A(a_{12}), i_A(a_{12}), f_A(a_{12})\right) \end{matrix}\right| .$$

$$\left|\begin{matrix} \left(t_A(a_{21}), i_A(a_{21}), f_A(a_{21})\right) & \left(t_A(a_{22}), i_A(a_{22}), f_A(a_{22})\right) \\ \left(t_A(a_{21}), i_A(a_{21}), f_A(a_{21})\right) & \left(t_A(a_{22}), i_A(a_{22}), f_A(a_{22})\right) \end{matrix}\right| \leq |A|.$$

Proof

We have that

$$\left|\begin{matrix} \left(t_A(a_{11}), i_A(a_{11}), f_A(a_{11})\right) & \left(t_A(a_{12}), i_A(a_{12}), f_A(a_{12})\right) \\ \left(t_A(a_{11}), i_A(a_{11}), f_A(a_{21})\right) & \left(t_A(a_{12}), i_A(a_{12}), f_A(a_{12})\right) \end{matrix}\right| .$$

$$. \left|\begin{matrix} \left(t_A(a_{21}), i_A(a_{21}), f_A(a_{21})\right) & \left(t_A(a_{22}), i_A(a_{22}), f_A(a_{22})\right) \\ \left(t_A(a_{21}), i_A(a_{21}), f_A(a_{21})\right) & \left(t_A(a_{22}), i_A(a_{22}), f_A(a_{22})\right) \end{matrix}\right|$$

$$\leq \left(\left(t_A(a_{11}), i_A(a_{11}), f_A(a_{11})\right). \left(t_A(a_{12}), i_A(a_{12}), f_A(a_{12})\right)\right).$$

$$. \left(\left(t_A(a_{21}), i_A(a_{21}), f_A(a_{21})\right). \left(t_A(a_{22}), i_A(a_{22}), f_A(a_{22})\right)\right)$$

$$\leq \left(t_A(a_{11}), i_A(a_{11}), f_A(a_{11})\right). \left(t_A(a_{22}), i_A(a_{22}), f_A(a_{22})\right)$$

$$+ \left(t_A(a_{12}), i_A(a_{12}), f_A(a_{12})\right). \left(t_A(a_{21}), i_A(a_{21}), f_A(a_{21})\right) = |A|$$

4.15. Definition:

Let A be a neutrosophic matrix of order $n \times n$. The matrix $A_{(p \to q)}$ is the neutrosophic matrix that is a result of A by replacing the row q by the row p from A.

4.16. Theorem:

Let A be a neutrosophic matrix of order $n \times n$. Then

1) $\left|A_{(1\to 2)}\right| . \left|A_{(2\to 1)}\right| \leq |A|$,
2) $\left|A_{(q\to p)}\right| . \left|A_{(p\to k)}\right| \leq |A|$.

Proof

1) By the theorem 4.12, we can write:

$$\left|A_{(1\to 2)}\right| . \left|A_{(2\to 1)}\right|$$

$$= \left(\sum_{p<q} \left|\begin{matrix} \left(t_A(a_{1p}), i_A(a_{1p}), f_A(a_{1p})\right) & \left(t_A(a_{1q}), i_A(a_{1q}), f_A(a_{1q})\right) \\ \left(t_A(a_{1p}), i_A(a_{1p}), f_A(a_{1p})\right) & \left(t_A(a_{1q}), i_A(a_{1q}), f_A(a_{1q})\right) \end{matrix}\right| . \left|A_{\binom{1\ 2}{p\ q}}\right| \right).$$

$$. \left(\sum_{r<s} \left|\begin{matrix} \left(t_A(a_{2r}), i_A(a_{2r}), f_A(a_{2r})\right) & \left(t_A(a_{2s}), i_A(a_{2s}), f_A(a_{2s})\right) \\ \left(t_A(a_{2r}), i_A(a_{2r}), f_A(a_{2r})\right) & \left(t_A(a_{2s}), i_A(a_{2s}), f_A(a_{2s})\right) \end{matrix}\right| . \left|A_{\binom{1\ 2}{r\ s}}\right| \right)$$

$$\leq \sum_{\substack{p<q \\ r<s}} \left|\begin{matrix} \left(t_A(a_{1p}), i_A(a_{1p}), f_A(a_{1p})\right) & \left(t_A(a_{1q}), i_A(a_{1q}), f_A(a_{1q})\right) \\ \left(t_A(a_{2r}), i_A(a_{2r}), f_A(a_{2r})\right) & \left(t_A(a_{2s}), i_A(a_{2s}), f_A(a_{2s})\right) \end{matrix}\right| . \left|A_{\binom{1\ 2}{p\ q}}\right| . \left|A_{\binom{1\ 2}{r\ s}}\right|$$

$$= \sum_{\substack{p<q \\ r<s \\ (p,q)\neq(r,s)}} \left|\begin{matrix} \left(t_A(a_{1p}), i_A(a_{1p}), f_A(a_{1p})\right) & \left(t_A(a_{1q}), i_A(a_{1q}), f_A(a_{1q})\right) \\ \left(t_A(a_{2r}), i_A(a_{2r}), f_A(a_{2r})\right) & \left(t_A(a_{2s}), i_A(a_{2s}), f_A(a_{2s})\right) \end{matrix}\right| . \left|A_{\binom{1\ 2}{p\ q}}\right| . \left|A_{\binom{1\ 2}{r\ s}}\right|$$

$$+ \sum_{p<q} \left|\begin{matrix} \left(t_A(a_{1p}), i_A(a_{1p}), f_A(a_{1p})\right) & \left(t_A(a_{1q}), i_A(a_{1q}), f_A(a_{1q})\right) \\ \left(t_A(a_{2p}), i_A(a_{2p}), f_A(a_{2p})\right) & \left(t_A(a_{2q}), i_A(a_{2q}), f_A(a_{2q})\right) \end{matrix}\right| . \left|A_{\binom{1\ 2}{p\ q}}\right|$$

By the theorem 4.12, we know

$$\sum_{p<q} \left|\begin{matrix} \left(t_A(a_{1p}), i_A(a_{1p}), f_A(a_{1p})\right) & \left(t_A(a_{1q}), i_A(a_{1q}), f_A(a_{1q})\right) \\ \left(t_A(a_{2p}), i_A(a_{2p}), f_A(a_{2p})\right) & \left(t_A(a_{2q}), i_A(a_{2q}), f_A(a_{2q})\right) \end{matrix}\right| . \left|A_{\binom{1\ 2}{p\ q}}\right| = |A|$$

If $(p,q) = (1,2)$ & $(r,s) = (1,3)$, then

$$\left|\begin{matrix} \left(t_A(a_{11}), i_A(a_{11}), f_A(a_{11})\right) & \left(t_A(a_{12}), i_A(a_{12}), f_A(a_{12})\right) \\ \left(t_A(a_{21}), i_A(a_{21}), f_A(a_{21})\right) & \left(t_A(a_{23}), i_A(a_{23}), f_A(a_{23})\right) \end{matrix}\right| . \left|A_{\binom{1\ 2}{1\ 2}}\right| . \left|A_{\binom{1\ 2}{1\ 3}}\right|$$

$= \Big(\big(t_A(a_{11}), i_A(a_{11}), f_A(a_{11})\big).\big(t_A(a_{23}), i_A(a_{23}), f_A(a_{23})\big)$

$+ \big(t_A(a_{12}), i_A(a_{12}), f_A(a_{12})\big).\big(t_A(a_{21}), i_A(a_{21}), f_A(a_{21})\big)\Big) . \left|A_{\binom{1\ 2}{1\ 2}}\right| . \left|A_{\binom{1\ 2}{1\ 3}}\right|$

$= \big(t_A(a_{11}), i_A(a_{11}), f_A(a_{11})\big).\big(t_A(a_{23}), i_A(a_{23}), f_A(a_{23})\big). \left|A_{\binom{1\ 2}{1\ 2}}\right| \left|A_{\binom{1\ 2}{1\ 3}}\right|$

$+ \big(t_A(a_{12}), i_A(a_{12}), f_A(a_{12})\big).\big(t_A(a_{21}), i_A(a_{21}), f_A(a_{21})\big). \left|A_{\binom{1\ 2}{1\ 2}}\right| . \left|A_{\binom{1\ 2}{1\ 3}}\right|$

$\leq \big(t_A(a_{11}), i_A(a_{11}), f_A(a_{11})\big).\big(t_A(a_{23}), i_A(a_{23}), f_A(a_{23})\big). \left|A_{\binom{1\ 2}{1\ 3}}\right|$

$+ \big(t_A(a_{12}), i_A(a_{12}), f_A(a_{12})\big).\big(t_A(a_{21}), i_A(a_{21}), f_A(a_{21})\big). \left|A_{\binom{1\ 2}{1\ 2}}\right|$

$\leq \begin{vmatrix} \big(t_A(a_{11}), i_A(a_{11}), f_A(a_{11})\big) & \big(t_A(a_{13}), i_A(a_{13}), f_A(a_{13})\big) \\ \big(t_A(a_{21}), i_A(a_{21}), f_A(a_{21})\big) & \big(t_A(a_{23}), i_A(a_{23}), f_A(a_{23})\big) \end{vmatrix} . \left|A_{\binom{1\ 2}{1\ 3}}\right|$

$+ \begin{vmatrix} \big(t_A(a_{11}), i_A(a_{11}), f_A(a_{11})\big) & \big(t_A(a_{12}), i_A(a_{12}), f_A(a_{12})\big) \\ \big(t_A(a_{21}), i_A(a_{21}), f_A(a_{21})\big) & \big(t_A(a_{22}), i_A(a_{22}), f_A(a_{22})\big) \end{vmatrix} . \left|A_{\binom{1\ 2}{1\ 2}}\right|$

$\leq |A| + |A| = |A|$.

Considering all the coordinates $(r,s) \neq (1,2)$ involved in (1,3) and $(n-1,n)$, we obtain that,

$$\begin{vmatrix} \big(t_A(a_{11}), i_A(a_{11}), f_A(a_{11})\big) & \big(t_A(a_{12}), i_A(a_{12}), f_A(a_{12})\big) \\ \big(t_A(a_{2r}), i_A(a_{2r}), f_A(a_{2r})\big) & \big(t_A(a_{2s}), i_A(a_{2s}), f_A(a_{2s})\big) \end{vmatrix}.$$

$$\left|A_{\binom{1\ 2}{1\ 2}}\right| . \left|A_{\binom{1\ 2}{r\ s}}\right| \leq |A|.$$

If we apply a same argument for any $p, q \in \{1,2,\dots,n\}$; $p < q$,

this completes the Proof.

2) The Proof as (1).

4.17. Example:

Let $A = \begin{bmatrix} (0.7,0.2,0.1) & (0.2,0.2,0.5) \\ (0.25,0.11,0.2) & (0.1,0,0.14) \end{bmatrix}$. Then

$A_{(1\to 2)} = \begin{bmatrix} (0.7,0.2,0.1) & (0.2,0.2,0.5) \\ (0.7,0.2,0.1) & (0.2,0.2,0.5) \end{bmatrix}$ &

$A_{(2\to 1)} = \begin{bmatrix} (0.25,0.11,0.2) & (0.1,0,0.14) \\ (0.25,0.11,0.2) & (0.1,0,0.14) \end{bmatrix}$.

We have $|A_{(2\to 1)}| = (0.1,0,0.2)$, $|A_{(1\to 2)}| = (0.2,0.2,0.5)$ and $|A| = (0.2,0.11,0.14)$.

Therefore, $|A_{(1\to 2)}|.|A_{(2\to 1)}| = (0.1,0,0.5) \leq |A|$.

4.18. Example:

Let $A = \begin{bmatrix} (0.3,0.1,0.2) & (0.11,0.45,0.23) & (0.33,0.5,0.1) \\ (0.22,0.4,0.16) & (0.4,0.2,0.1) & (0.7,0.2,0.1) \\ (0.9,0.1,0.2) & (0,0.1,0.2) & (0.15,0.4,0.9) \end{bmatrix}$. Then

$A_{(1\to 2)} = \begin{bmatrix} (0.3,0.1,0.2) & (0.11,0.45,0.23) & (0.33,0.5,0.1) \\ (0.3,0.1,0.2) & (0.11,0.45,0.23) & (0.33,0.5,0.1) \\ (0.9,0.1,0.2) & (0,0.1,0.2) & (0.15,0.4,0.9) \end{bmatrix}$ &

$A_{(2\to 3)} = \begin{bmatrix} (0.3,0.1,0.2) & (0.11,0.45,0.23) & (0.33,0.5,0.1) \\ (0.22,0.4,0.16) & (0.4,0.2,0.1) & (0.7,0.2,0.1) \\ (0.22,0.4,0.16) & (0.4,0.2,0.1) & (0.7,0.2,0.1) \end{bmatrix}$

We have $|A_{(2\to 3)}| = (0.3,0.2,0.16)$, $|A_{(1\to 2)}| = (0.11,0.2,0.2)$ and $|A| = (0.33,0.4,0.2)$.

Therefore, $|A_{(2\to 3)}|.|A_{(1\to 2)}| = (0.11,0.1,0.2) \leq |A|$.

4.19. Remark:

If A is a classical matrix of order $n \times n$, then we know that $|A| = 0$, when the row p equals to the row q $(p \neq q)$. But this problem in neutrosophic matrices is different, as in the previous examples.

4.20. Theorem:

Let A be a neutrosophic matrix of order $n \times n$. Then

$|A.adj(A)| = |A| = |adj(A).A|$.

Proof

We prove that $|A.adj(A)| = |A|$.

First, we consider $n = 2$:

Let $A = \begin{bmatrix} \big(t_A(a_{11}), i_A(a_{11}), f_A(a_{11})\big) & \big(t_A(a_{12}), i_A(a_{12}), f_A(a_{12})\big) \\ \big(t_A(a_{21}), i_A(a_{21}), f_A(a_{21})\big) & \big(t_A(a_{22}), i_A(a_{22}), f_A(a_{22})\big) \end{bmatrix}$.

Thus,

$adj(A) = \begin{bmatrix} \big(t_A(a_{22}), i_A(a_{22}), f_A(a_{22})\big) & \big(t_A(a_{12}), i_A(a_{12}), f_A(a_{12})\big) \\ \big(t_A(a_{21}), i_A(a_{21}), f_A(a_{21})\big) & \big(t_A(a_{11}), i_A(a_{11}), f_A(a_{11})\big) \end{bmatrix}$

$\Longrightarrow \ A.adj(A) =$

$\begin{bmatrix} |A| & \big(t_A(a_{11}), i_A(a_{11}), f_A(a_{11})\big).\big(t_A(a_{12}), i_A(a_{12}), f_A(a_{12})\big) \\ \big(t_A(a_{21}), i_A(a_{21}), f_A(a_{21})\big).\big(t_A(a_{22}), i_A(a_{22}), f_A(a_{22})\big) & |A| \end{bmatrix}$

$\Longrightarrow \ |A.adj(A)|$

$= |A| + \big(t_A(a_{11}), i_A(a_{11}), f_A(a_{11})\big).\big(t_A(a_{12}), i_A(a_{12}), f_A(a_{12})\big).$

$.\big(t_A(a_{21}), i_A(a_{21}), f_A(a_{21})\big).\big(t_A(a_{22}), i_A(a_{22}), f_A(a_{22})\big) = |A|$

Next, we consider $n > 2$, we have

$A.adj(A) =$

$$\begin{bmatrix} \sum_{t=1}^{n}\big(t_A(a_{1t}), i_A(a_{1t}), f_A(a_{1t})\big) . |A_{1t}| & \dots & \sum_{t=1}^{n}\big(t_A(a_{1t}), i_A(a_{1t}), f_A(a_{1t})\big) . |A_{nt}| \\ \vdots & \ddots & \vdots \\ \sum_{t=1}^{n}\big(t_A(a_{nt}), i_A(a_{nt}), f_A(a_{nt})\big) . |A_{1t}| & \dots & \sum_{t=1}^{n}\big(t_A(a_{nt}), i_A(a_{nt}), f_A(a_{nt})\big) . |A_{nt}| \end{bmatrix}$$

$$= \left[\sum_{t=1}^{n}\big(t_A(a_{it}), i_A(a_{it}), f_A(a_{it})\big) . |A_{jt}|\right]$$

$$\Longrightarrow \ |A.adj(A)| = \sum_{\sigma\in S_n}\prod_{i=1}^{n}\left(\sum_{t=1}^{n}\big(t_A(a_{it}), i_A(a_{it}), f_A(a_{it})\big) . |A_{\sigma(i)t}|\right)$$

(1) If $\sigma = e$, where e is the identity of the group S_n, then

$\prod_{i=1}^{n}\left(\sum_{t=1}^{n}\big(t_A(a_{it}), i_A(a_{it}), f_A(a_{it})\big) . |A_{\sigma(i)t}|\right) = |A|$.

(2) Suppose that there exists $k \in \{1,2,\dots,n\}$ such that $\sigma(k) = k$. Then

$$\sum_{t=1}^{n}\big(t_A(a_{kt}), i_A(a_{kt}), f_A(a_{kt})\big) . |A_{\sigma(k)t}| = \sum_{t=1}^{n}\big(t_A(a_{kt}), i_A(a_{kt}), f_A(a_{kt})\big) . |A_{kt}| = |A|.$$

and

$$\prod_{i=1}^{n}\left(\sum_{t=1}^{n}\big(t_A(a_{it}), i_A(a_{it}), f_A(a_{it})\big) . |A_{\sigma(i)t}|\right)$$

$$= \left(\sum_{t=1}^{n}\big(t_A(a_{1t}), i_A(a_{1t}), f_A(a_{1t})\big) . |A_{\sigma(1)t}|\right) \dots |A| \dots \left(\sum_{t=1}^{n}\big(t_A(a_{nt}), i_A(a_{nt}), f_A(a_{nt})\big) . |A_{\sigma(n)t}|\right)$$

$\leq |A|$.

(3) Assume that $\sigma(k) \neq k$ for all $k \in \{1,2,\dots,n\}$. The permutation σ can be written as $\sigma = \sigma_1\,\sigma_2 \dots \sigma_s$; $\sigma_1, \sigma_2, \dots, \sigma_s$ are disjoint cycles.

If $\sigma_1 = (1\ 2)$, we have

$$\left(\sum_{t=1}^{n}\big(t_A(a_{1t}), i_A(a_{1t}), f_A(a_{1t})\big) . |A_{\sigma(1)t}|\right).$$

$$\left(\sum_{t=1}^{n}\left(t_A(a_{2t}), i_A(a_{2t}), f_A(a_{2t})\right).\left|A_{\sigma(2)t}\right|\right)$$
$$= \left(\sum_{t=1}^{n}\left(t_A(a_{1t}), i_A(a_{1t}), f_A(a_{1t})\right).\left|A_{2t}\right|\right).$$
$$.\left(\sum_{t=1}^{n}\left(t_A(a_{2t}), i_A(a_{2t}), f_A(a_{2t})\right).\left|A_{1t}\right|\right) = \left|A_{(1\to 2)}\right|.\left|A_{(2\to 1)}\right| \leq |A|$$

By the same argument, if $\sigma_1 = (p \;\; q)$, where $p, q \in \{1,2,\dots,n\}$, we can prove that

$$\prod_{i=1}^{n}\left(\sum_{t=1}^{n}\left(t_A(a_{it}), i_A(a_{it}), f_A(a_{it})\right).\left|A_{\sigma(i)t}\right|\right) \leq |A|$$

If $\sigma_1 = (p_1 \;\; p_2 \dots p_r)$, where $p_1, p_2, \dots, p_r \in \{1,2,\dots,n\}$, then we see that

$$\left(\sum_{t=1}^{n}\left(t_A(a_{p_1t}), i_A(a_{p_1t}), f_A(a_{p_1t})\right).\left|A_{\sigma(p_1)t}\right|\right).\dots$$
$$.\left(\sum_{t=1}^{n}\left(t_A(a_{p_rt}), i_A(a_{p_rt}), f_A(a_{p_rt})\right).\left|A_{\sigma(p_r)t}\right|\right)$$
$$= \left(\sum_{t=1}^{n}\left(t_A(a_{p_1t}), i_A(a_{p_1t}), f_A(a_{p_1t})\right).\left|A_{p_2t}\right|\right).\dots$$
$$.\left(\sum_{t=1}^{n}\left(t_A(a_{p_rt}), i_A(a_{p_rt}), f_A(a_{p_rt})\right).\left|A_{p_1t}\right|\right)$$
$$= \left|A_{(p_2\to p_1)}\right|.\left|A_{(p_3\to p_2)}\right| \dots \left|A_{(p_1\to p_r)}\right| \leq |A|$$; by theorem 4.16.

According to the above discussions, for any $\sigma \in S_n$, we obtain that

$$\prod_{i=1}^{n}\left(\sum_{t=1}^{n}\left(t_A(a_{it}), i_A(a_{it}), f_A(a_{it})\right).\left|A_{\sigma(i)t}\right|\right) \leq |A|$$
$$\Rightarrow |A.adj(A)| = |A|.$$

Similarly, we can prove that $|adj(A).A| = |A|$.

5. Conclusion

In our work, we have studied some properties of the determinant of a neutrosophic matrix. Most of the proven properties are similar to the properties of the determinant of a classical matrix. In addition, the important relationship $|A.adj(A)| = |A| = |adj(A).A|$ has been proven.

Biographies

Eman Alabdullah

Department of Mathematics, Faculty of Sciences, Aleppo University, Aleppo, Syria, 00963988113280, emankhder1987@gmail.com

Eman is from Deir-ez-Zor, Syria. She is a teaching assistant in the Department of Mathematics, Faculty of Sciences at Al Furat University in Syria. Her specialty is linear algebra. Eman received an MSc degree in Mathematics from Aleppo University in 2021 for research on neutrosophic modules. She published two research papers in the Research Journal of Aleppo University. Eman is currently a postgraduate (PhD) in the Department of Mathematics, Faculty of Sciences at Aleppo University. Her main research interests include linear algebra, module theory, number theory, and integral equations.

ORCID: 0009-0002-1874-6049

Safwan Aouira

Department of Mathematics, Faculty of Science, Aleppo University, Aleppo, Syria, 00963992135937, safwanaouira@gmail.com

Prof Aouira is from Aleppo, Syria. He is a professor in the Department of Mathematics, Faculty of Sciences at Aleppo University in Syria. He received a PhD from Humboldt University of Berlin in 1990. Prof Aouira served as a lecturer at King Faisal University and Dammam University in the Kingdom of Saudi Arabia (2006–2011). He is an author of several university books in Syria and the Kingdom of Saudi Arabia. He has published papers in peer-reviewed scientific journals. His main research interests include algebra and algebraic geometry.

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