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Perfect Roman and Perfect Italian Domination of Cartesian Product Graphs

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ABSTRACT

For a graph $G = (V, E)$, a function $f: V \to \{0,1,2\}$ is a perfect Roman dominating function (PRDF) on G if every $v \in V$ with $f(v) = 0$ is adjacent to exactly one vertex u with $f(u) = 2$. The sum $\sum_{v\in V} f(v)$ is the weight $w(f)$ of f. The perfect Roman domination number $\gamma_R^p(G)$ of G is least positive integer k such that there is a PRDF f on G with $w(f) \le k$. A function $f: V \to \{0,1,2\}$ is a perfect Italian dominating function (PIDF) on G if for every $v \in V$ with $f(v) = 0$, $\sum_{u\in N(v)} f(u) = 2$. The sum $\sum_{v\in V} f(v)$ is the weight $w(f)$. The perfect Italian domination number $\gamma_I^p(G)$ of G is least positive integer k such that there is a PIDF f on G with $w(f) \le k$. Perfect Roman domination and perfect Italian domination are variants of Roman domination, which was originally introduced as a defensive strategy of the Roman Empire. In this article, we prove that the perfect Roman domination and perfect Italian domination problems for Cartesian product graphs are NP-complete. We also give an upper bound for $\gamma_I^p(G)$, where G is the Cartesian product of paths and cycles.

KEYWORDS
graph domination, graph operations, graph theory, np-completeness, problem complexity, simple graphs

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1. Introduction and Preliminaries

In this paper, we consider finite, simple and undirected graphs $G = (V, E)$, with vertex set V and edge set E. We denote the number of vertices in a graph G by $|G|$. Two vertices, u and v, in G are *adjacent* or *neighbours* if $uv \in E$. The set of neighbours of a vertex v in G is denoted by $N_G(v)$ or by $N(v)$ if G is known from the context. The number of edges in a path is its *length*. We denote a path with k vertices by P_k. A cycle graph with k vertices is denoted by C_k.

Roman domination was first introduced by Cockayne *et al.,* (2004) after a series of papers on strategies used to defend the ancient Roman Empire (ReVelle, 1997; Stewart, 1999; Revelle and Rosing, 2000). The Roman domination notion was inspired by Emperor Constantine's (272–337 AD) defence plan to protect the Roman Empire. The approach was as follows: (i) any city in the empire could have no more than two legions stationed there, and (ii) every city without a legion had to be near a city with two armies. Therefore, if an attack were launched against a city without an army, a city with two armies could send one of its armies to defend the former. Roman domination and its variants have been the subject of more than 100 academic articles. Even though the original strategy focused on army distribution, it can now be applied to any distribution problem, such as service centre distribution. Perfect Roman domination introduced in (Henning *et al.,* 2018), Italian domination introduced as Roman $\{2\}$-domination in (Chellali *et al.,* 2016) and perfect Italian domination introduced in (Haynes and Henning, 2019) are variants of Roman domination. In the current paper, we continue the study of perfect Roman and perfect Italian dominations.

For a graph $G = (V, E)$, every function $f: V \to A$, where $A \subset \mathbb{Z}$, corresponds to the partition $\left(V_i^f \middle| V_i^f := \{v \in V | f(v) = i\}, i \in A\right)$. The weight of f is $w(f) := \sum_{v\in V} f(v)$. If H is a subgraph of G, then $w(f(H)) := \sum_{v\in V(H)} f(v)$. A Roman dominating function (RDF) is a function $f: V \to \{0,1,2\}$ such that every $v \in V_0^f$ is adjacent to at least one vertex in V_2^f. The Roman domination number of G, denoted by $\gamma_R(G)$, is the minimum weight of an RDF f on G. For recent work in Roman domination, we refer the reader to Luiz (2024). A function $f: V(G) \to \{0,1,2\}$ is a *perfect RDF* (PRDF) on G if every $v \in V_0^f$ is adjacent to exactly one vertex $u \in V_2^f$. The *perfect Roman domination number* of a graph G, denoted by $\gamma_R^p(G)$, is the minimum weight of a PRDF on G. It is clear that any PRDF on G is also an RDF, so $\gamma_R(G) \le \gamma_R^p(G)$ for every graph G. We refer the reader to (Henning and Klostermeyer, 2018; Darkooti *et al.*, 2019 and Cabrera Martínez, 2022) for further work in perfect Roman domination.

We now provide a simple example to illustrate the concept of perfect Roman domination: Let $P_5 := v_1v_2v_3v_4v_5$ be a path with five vertices. Let $f: V(P_5) \to \{0,1,2\}$ be a function defined as $f(v_1) = 0$, $f(v_2) = 2$, $f(v_3) = 2, f(v_4) = 0, f(v_5) = 1$ is a PRDF, as every vertex with weight equal to 0 (namely, v_1 and v_4) is adjacent to a vertex with weight equal to 2. Observe that $w(f) = 5$ but that $\gamma_R^p(G) \ne 5$, as we can find another PRDF with a weight less than 5. Define a function $g: V(P_5) \to \{0,1,2\}$ such that $g(v_1) = 2$, $g(v_2) = 0$, $g(v_3) = 0, g(v_4) = 2$ and $g(v_5) = 0$. Then, g is a PRDF on P_5 with $w(g) = 4$. It is not difficult to check that there is no PRDF on P_5 with a weight less than 4. So, $\gamma_R^p(P_5) = 4$.

An *Italian dominating function* (IDF) on G is a function $f: V(G) \to \{0,1,2\}$ such that if $v \in V_0^f$, $\sum_{u\in N(v)} f(u) \ge 2$. The minimum weight of an IDF on G is called the *Italian domination number* of G, denoted by $\gamma_I(G)$. Observe that every RDF on a graph G is also an IDF, so $\gamma_I(G) \le \gamma_R(G) \le \gamma_R^p(G)$. Italian domination is also called Roman $\{2\}$-domination in the literature. We refer the reader to (Almulhim *et al.*, 2024) for a recent survey paper on Roman $\{2\}$-domination.

A function $f: V(G) \to \{0,1,2\}$ is a *perfect IDF* (PIDF) on G if for every $v \in V_0^f$, $\sum_{u\in N(v)} f(u) = 2$. The *perfect Italian domination number* of G, denoted by $\gamma_I^p(G)$, is the minimum weight of a PIDF on G. Clearly, every PIDF on G is also an IDF, so $\gamma_I(G) \le \gamma_I^p(G)$. To read more about perfect Italian domination, we refer the reader to (Nazari-Moghaddam and Chellali, 2022; Banerjee *et al.*, 2021; Pradhan *et al.*, 2022).

To illustrate the concept of perfect Italian domination, let $f: V(P_5) \to \{0,1,2\}$ be a function on P_5 such that $f(v_1) = 1$, $f(v_2) = 0$, $f(v_3) = 1, f(v_4) = 0$ and $f(v_5) = 1$. Then, $V_0^f = \{v_2, v_4\}$, $\sum_{u\in N(v_2)} f(u) = f(v_1) + f(v_3) = 2$ and $\sum_{u\in N(v_4)} f(u) = f(v_3) + f(v_5) = 2$. So f is a PIDF on P_5 with $w(f) = 3$. It is not

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difficult to check that there is no PIDF on P_5 with a weight less than 3. So, $\gamma_I^p(P_5) = 3$.

In general, none of the numbers $\gamma_R(G)$ and $\gamma_I^p(G)$ is bound for the other. Let $G = K_{n_1,n_2,\cdots,n_m}$ be the complete m-partite graph where $n_i \geq 3$ for each $i \in [m]$. If $m \geq 4$, $\gamma_I^p(G) = n_1 + n_2 + \cdots + n_m$ (Lauri and Mitillos, 2020); while, $\gamma_R(G) \leq 4$, as we can label one vertex from the first partite and one vertex from the second partite with 2 and label the rest of the vertices with 0. If $m = 3$, $\gamma_I^p(G) = 3$ (Lauri and Mitillos, 2020), while $\gamma_R(G) = 4$ (Cockayne *et al.*, 2004). As $\gamma_R(G) \leq \gamma_R^p(G)$, the latest example yields a graph G with $\gamma_I^p(G) \leq \gamma_R^p(G)$. Generally, none of the numbers $\gamma_R^p(G)$ and $\gamma_I^p(G)$ is bound for the other. If $m \geq 4$, $\gamma_R^p(G) \leq n_\alpha + 1$, where $n_\alpha = \min_{i\in[m]}\{n_i\}$, as we can label one vertex in the n_α-partite with 2, label the rest of vertices in the same partite with 1 and label the rest of vertices in G with 0. So, $\gamma_R^p(G) \leq \gamma_I^p(G)$.

This paper aims to increase the list of NP-complete problems. In mathematics, the NP-completeness theory is used to determine how difficult it is to find a polynomial time algorithm to solve a decision problem. More than 3,000 problems in graph theory and computer science have proven to be NP-complete problems when expressed as decision problems. This list is growing rapidly. All NP-complete problems are thought to have similar hardness. So, if one of those problems is solved in polynomial time, all other NP-complete problems can be solved in polynomial time. This observation hints at the usefulness of having a large list of NP-complete problems.

Darkooti *et al.,* (2019) proved that perfect Roman domination is an NP-complete problem for bipartite graphs. In the current paper, similar methods are used to prove that the perfect Roman domination problem is NP-complete for Cartesian product graphs.

Lauri and Mitillos (2020) proved that perfect Italian domination is NP-complete even if G is a bipartite planar graph. In this paper, we follow the technique they used and prove that perfect Italian domination is NP-complete even if G is a Cartesian product graph.

In the last section of this paper, we discuss perfect Italian domination of the Cartesian product graph of the paths P_r and P_s, or of the path P_r and the cycle C_s, or of the cycles C_r and C_s, where $r, s \geq 6$. We also give un upper bound for $\gamma_I^p(G)$, where G is a Cartesian product of graphs. For perfect Roman domination of the Cartesian product of cycles and paths, we refer the readers to Almulhim *et al.,* (2022).

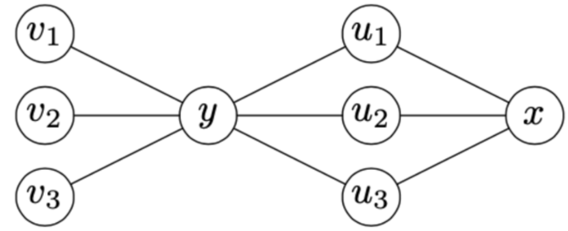
Let G_1 and G_2 be two graphs. The *Cartesian product graph* of G_1 and G_2, denoted by $G_1 \boxtimes G_2$, is a graph with the Cartesian product $V(G_1) \times V(G_2)$ as its set of vertices. Two vertices $(v,u), (v',u') \in G_1 \boxtimes G_2$ are adjacent if either

$v = v'$ and $uu' \in G_2$ or

$u = u'$ and $vv' \in G_1$.

To prove our results, we must first introduce several graphs. Let $H_{x,l} := \left(V(H_{x,l}), E(H_{x,l})\right)$, $l \geq 1$, where $V(H_{x,l}) := \{x, y, u_1, \cdots, u_l, v_1, \cdots, v_l\}$ and $E(H_{x,l}) := \{xu_i, yu_i, yv_i \mid i \in [l]\}$. The graph $H_{x,3}$ is shown in Figure 1. Let $H_c := \left(V(H_c), E(H_c)\right)$, where $V(H_c) := \{c, c', c''\}$ and $E(H_c) := \{cc', cc''\}$, so H_c is a path of length two. Let $O_c := \left(V(O_c), E(O_c)\right)$, where $V(O_c) := \{c, c', c'', d', d''\}$ and $E(O_c) := \{cc', cc'', cd', cd'', c'c'', d'd''\}$. Let K_2 be a clique of size two with the vertex set $\{w_1, w_2\}$. Let $F_{x,l} := H_{x,l} \boxtimes K_2$, $F_c := H_c \boxtimes K_2$ and $Q_c := O_c \boxtimes K_2$.

Figure 1. The graph $\mathbf{H_{x,3}}$.



2. The Complexity of Perfect Roman Domination of Cartesian Product Graphs

In this section, we prove that the perfect Roman domination problem for Cartesian product graphs is NP-complete.

To begin, we write the problem as a decision problem, that is, a problem for which the answer to its instances is either yes or no. So, instead of asking, 'Let G be a graph. Find a PRDF on G of minimum weight', we let PERFECT ROMAN DOMINATION be the decision problem where a graph G and an integer k are given. The goal is to decide whether G has a PRDF of weight at most k.

If a function $f: V(G) \to \{0,1,2\}$ is given, we can check in polynomial time (with respect to the number of vertices in the graph) whether the function is PRDF of weight less than or equal to k. We need to check only the weight of the neighbours of each vertex and whether the sum of all vertices' weight is at most k. Thus, the problem is NP. We proceed by providing a polynomial-time reduction from the decision problem EXACT 3-COVER (X3C). (X3C). What we mean by a polynomial-time reduction is that if there is a polynomial algorithm to solve the PERFECT ROMAN DOMINATION problem, then there is a polynomial algorithm to solve the X3C problem. So, the PERFECT ROMAN DOMINATION problem is not more difficult than the X3C problem.

In the X3C problem, a set X with $|X| = 3q$ and a collection C of 3-element subsets of X are given. The question is, does C contain a subset C' such that every element $x \in X$ is in precisely one element of C'? If the answer is yes, we say (X, C) has an *exact cover*. The X3C problem is known to be NP-complete (Johnson and Garey, 1979).

Proposition 1. Let f be a PRDF on G containing $F_{x,l}$ as a subgraph, with $\{(x,w_1),(x,w_2)\}$ as a vertex cut. If $f(x,w_1) = 2$, then $w\left(f(F_{x,l})\right) \geq l + 2$.

Proof. Assume that $f(y,w_1) = 2$. Therefore, for all $i \in [l]$, the vertex (u_i,w_1) is adjacent to two vertices labelled 2. So, we must have $f(u_i,w_1) \geq 1$ for all $i \in [l]$. This implies that $w\left(f(F_{x,l})\right) \geq l + 4$. So, the statement holds. We may assume that $f(y,w_1) \neq 2$. If $f(v_i,w_1) = 0$ for some $i \in [l]$, then we must have $f(v_i,w_2) = 2$. So, for all $i \in [l]$, either $f(v_i,w_1) \geq 1$ or $f(v_i,w_2) \geq 1$. This implies that $w\left(f(F_{x,l})\right) \geq l + 2$. Thus, the statement holds. ■

Remark 1. *Due to the symmetry between the vertices* (x,w_1) *and* (x,w_2), *the result of Proposition 1 holds if we write '*$f(x,w_2) = 2$*' instead of* $f(x,w_1) = 2$*'.*

Proposition 2. *Let* f *be a PRDF on a graph* G *containing* $F_{x,l}$, $l \geq 3$ *as a subgraph, with* $\{(x,w_1),(x,w_2)\}$ *as a vertex cut. If* $f(x,w_1) = 1$, *then* $w\left(f(F_{x,l})\right) \geq 5$.

Proof. We prove by discussing all possibilities of $f(x,w_2)$. If $f(x,w_2) = 2$, the result follows by Proposition 1 and Remark 1. If $f(x,w_2) = 1$, then clearly $w\left(f(F_{x,l})\right) \geq 6$. If $f(x,w_2) = 0$, it is clear that $w\left(f(F_{x,l})\right) \geq 5$. ■

Proposition 3. *Let f be a PRDF on a graph G containing $F_{x,l}$, $l \geq 3$ as a subgraph, with $\{(x,w_1),(x,w_2)\}$ as a vertex cut.*

If $f(x,w_1) = 0$ and $f(x,w_2) = 1$, then $w\left(f\left(F_{x,l}\right)\right) \geq 5$.

If $f(x,w_1) = 0$ and $f(x,w_2) = 0$, then $w\left(f\left(F_{x,l}\right)\right) \geq 4$.

If $w\left(f\left(F_{x,l}\right)\right) = 4$, then $f(y,w_1) = f(y,w_2) = 2$.

The following proposition is not difficult to check.

Proposition 4. *Let G be a graph containing F_c as a subgraph, with $\{(c,w_1),(c,w_2)\}$ as a vertex cut. If f is a PRDF on G, then f has allocated labels of weight greater than or equal to 4 to F_c.*

Theorem 1. PERFECT ROMAN DOMINATION *is NP-complete for Cartesian product graphs.*

Proof. Let (X,C) be an instance of $X3C$ such that $X = \{x_1, x_{2,} \cdots, x_{3q}\}$ and $C = \{C_1, C_2, \cdots, C_t\}$. We describe a polynomial-time reduction from the X3C instance to a PERFECT ROMAN DOMINATION instance.

Let H be the graph with vertex set $V(H) = \{x_1, x_{2,} \cdots, x_{3q}\} \cup \{c_1, c_2, \cdots, c_t\}$ such that $x_i c_j \in E(H)$ if and only if $x_i \in C_j$. Let $k = 12q + 4t$. Let Q be the graph obtained from H by identifying x_i for $i \in [3q]$ with x in $H_{x,k}$ (informally, for each $i \in [3q]$, attach a copy of $H_{x,k}$ to x_i) and attaching to c_i for $i \in [t]$ two pendants c'_i, c''_i (i.e. c_i is identified with c in H_c). We denote the vertex y in the graph $H_{x,k}$ that corresponds to x_i by y_i. Let G be the Cartesian product of Q and K_2 (with $\{w_1, w_2\}$ as the vertex set of K_2). We prove (X,C) has an exact cover if and only if G has a PRDF f with $w(f) \leq k$.

Let C' be an exact cover of (X,C). Define a function $f: V(G) \to \{0,1,2\}$ as follows: set $f(y_i, w_1) = f(y_i, w_2) = 2$ for all $i \in [3q]$; set $f(c_i, w_1) = f(c_i, w_2) = 2$ if $C_i \in C'$; set $f(c'_i, w_1) = f(c''_i, w_2) = 2$ if $C_i \in C \setminus C'$; and all the remaining vertices are labelled 0. As $|X| = 3q$ and $|C| = t$, then $w(f) = 4(3q) + 4t = 12q + 4t = k$. Since C' is an exact cover, every x_i is in exactly one 3-element subset $C_j \in C'$, so (x_i, w_1) is adjacent to exactly one vertex, namely (c_j, w_1), labelled 2. Similarly, (x_i, w_2) is adjacent to exactly one vertex, namely (c_j, w_2), labelled 2. It is not hard to see that any other vertex labelled 0 is adjacent to exactly one vertex labelled 2. Thus, f is a PRDF.

Conversely, assume that there exists a PRDF f on G such that $w(f) \leq k$. Note that G contains $3q$ copies of $F_{x,k}$. Observe also that for any $i \in [3q]$, $E\left(F_{x_i,k} - \{(x_i,w_1),(x_i,w_2)\}, G - F_{x_i,k}\right) = \phi$; informally, if $F_{x_i,k}$ is connected to $G - F_{x_i,k}$, it is connected only through the vertices (x_i,w_1) and (x_i,w_2). From Propositions 1, 2 and 3, f has allocated labels of weight greater than or equal to $12q$ to $\cup_{i\in[3q]} V\left(F_{x_i,k}\right)$. Note that G contains t copies of F_c. By Proposition 4, the function f has allocated labels of weight greater than or equal to $4t$ to $\cup_{i\in[t]} V\left(F_{c_i}\right)$. Therefore, f allocates labels of weight exactly $12q$ to $\cup_{i\in[3q]} V\left(F_{x_i,k}\right)$ and exactly $4t$ to $\cup_{i\in[t]} V\left(F_{c_i}\right)$. By Proposition 3, we must have $f(x_i,w_1) = f(x_i,w_2) = 0$ and $f(y_i,w_1) = f(y_i,w_2) = 2$ for all $i \in [3q]$. This also means that all neighbours of (x_i,w_1) in $F_{x_i,k}$ are labelled 0. Since f is a PRDF, (x_i,w_1) is adjacent to exactly one neighbour of the form (c_j,w_1) such that $f(c_j,w_1) = 2$. Thus $C' = \{C_j \mid f(c_j,w_1) = 2\}$ is an exact cover of (X,C). ■

3. The Complexity of Perfect Italian Domination of Cartesian Product Graphs

In this section, we prove that the perfect Italian domination problem of Cartesian product graphs is NP-complete.

Let the PERFECT ITALIAN DOMINATION be the decision problem where a graph G and an integer k are given and the goal is to decide whether G has a PIDF of weight at most k.

If a function $f: V(G) \to \{0,1,2\}$ is given, we can check in polynomial time whether the function is a PIDF with $w(f) \leq k$. Thus, the problem is in NP class. To proceed, a polynomial-time reduction from X3C will be given. We need some propositions before describing the reduction.

Proposition 5. *Let f be a PIDF on a graph G containing $F_{x,l}$ as subgraph, with $\{(x,w_1),(x,w_2)\}$ as a vertex cut. If $f(\mathrm{x},\mathrm{w}_1) \neq 0$, then $w\left(f\left(F_{x,l}\right)\right) > l$.*

Proof. Assume that $f(x,w_1) = 2$. If $f(u_i,w_1) > 0$ for all $i \in [l]$, *then* $w\left(f\left(F_{x,l}\right)\right) \geq l + 2$, and we are done. So, assume $f(u_i,w_1) = 0$ for some $i \in [l]$. Then, $f(y,w_1) = 0$. Let $Q := \{i \mid f(v_i,w_1) = 0\}$. Then, $f(v_i,w_2) = 2$ for all $i \in Q$. Thus, for every $i \in [l]$, either $f(v_i,w_1) \geq 1$ or $f(v_i,w_2) \geq 1$. *Then,* $w\left(f\left(F_{x,l}\right)\right) \geq l + 2$, and we are done.

Assume $f(x,w_1) = 1$. If $f(y,w_1) = 2$, then $f(u_i,w_1) > 0$ for all $i \in [l]$; so, $w\left(f\left(F_{x,l}\right)\right) \geq l + 3$, and we are done. If $f(y,w_1) = 1$, then for every $i \in [l]$, either $f(v_i,w_1) \geq 1$ or $f(v_i,w_2) \geq 1$. So, $w\left(f\left(F_{x,l}\right)\right) \geq l + 2$, and we are done. If $f(y,w_1) = 0$, then for every $i \in [l]$, either $f(u_i,w_1) > 0$ or $f(u_i,w_2) > 0$. So, $w\left(f\left(F_{x,l}\right)\right) \geq l + 1$, and we are done. ■

Remark 2. *Due to the symmetry between (x,w_1) and (x,w_2), the result of Proposition 5 holds if we write '$f(x,w_2) \neq 0$' instead of '$f(x,w_1) \neq 0$'.*

Let $u \in G$ and f be a PIDF on G. We say that u is *satisfied* if either $f(u) \neq 0$ or $\sum_{v\in N(u)} f(v) = 2$. Let $u \in H \subseteq G$ with $f(u) = 0$. We say that u is *in-satisfied* with respect to H if u is satisfied and $\sum_{v\in N(u)\cap H} f(v) = 2$; we say that u is *out-satisfied* with respect to H if u is satisfied and $\sum_{v\in N(u)\setminus H} f(v) = 2$.

Proposition 6. Let G be a graph *containing $F_{x,l}$, $l \geq 3$ as a subgraph, with $\{(x,w_1),(x,w_2)\}$ as a vertex cut. Let f be a PIDF on G. If $f(x,w_1) = 0$ and $f(x,w_2) = 0$, f has allocated labels of weight at least 4 to $F_{x,l}$, with equality if and only if $f(y,w_1) = f(y,w_2) = 2$.*

Proposition 7. *Let G be a graph containing Q_c as a subgraph, with $\{(c,w_1),(c,w_2)\}$ as a vertex cut. Let f be a PIDF on G. Then, $w\left(f(Q_c)\right) \geq 4$. In addition, if $w\left(f(Q_c)\right) = 4$, then either $f(c,w_1) = f(c,w_2) = 0$ or $f(c,w_1) = f(c,w_2) = 2$.*

Proof. If each vertex in the set $\{(c',w_1),(c'',w_1),(c',w_2),(c'',w_2)\}$ is labelled 1 or 2, *then* $w\left(f(Q_c)\right) \geq 4, and$ we are done. So, assume without losing generality that $f(c',w_1) = 0$. Since $N\left((c',w_1)\right) = \{(c,w_1),(c'',w_1),(c',w_2)\}$, the weight assigned to those vertices combined is 2. Note that no vertex in $N\left((c',w_1)\right)$ is adjacent to a vertex in $\{(d',w_2),(d'',w_2)\}$. If each vertex in $\{(d',w_2),(d'',w_2)\}$ is labelled 1 or 2, we are done. Without losing generality, suppose $f(d',w_2) = 0$. Since $N\left((d',w_2)\right) = \{(c,w_2),(d'',w_2),(d',w_1)\}$, the weight assigned to those vertices combined is 2. Thus, the first claim is true. The second claim is not difficult to check. ■

Theorem 2. *PERFECT ITALIAN DOMINATION is NP-complete for Cartesian product graphs.*

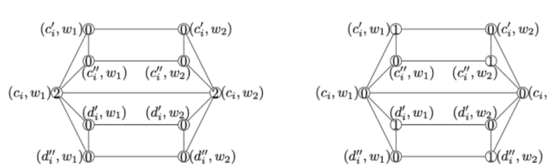
Proof. Let (X,C) be an instance of $X3C$ such that $X = \{x_1, x_{2,} \cdots, x_{3q}\}$ and $C = \{C_1, C_2, \cdots, C_t\}$ is a collection of 3-element subsets of X. We describe a polynomial-time reduction to *PERFECT ITALIAN DOMINATION.*

Let H be the graph with vertex set $V(H) = \{x_1, x_{2,} \cdots, x_{3q}\} \cup \{c_1, c_2, \cdots, c_t\}$, where $x_i c_j \in E(H)$ if and only if $x_i \in C_j$. Let $k = 12q + 4t$. Let Q be the graph acquired from H by identifying x_i for $i \in [3q]$ with x in $H_{x,k}$ (informally, for each $i \in [3q]$, attach a copy of $H_{x,k}$ to x_i) and identifying c_i with c in O_c (informally, for each $i \in [t]$, attach a copy of O_c to c_i). Let $G := Q \boxtimes K_2$, that is, G is the Cartesian product of Q and K_2. We prove that (X,C) has an exact cover if and only if G admits a PIDF f such that $w(f) \leq k$.

Assume that (X, C) has an exact cover C'. Define a function $f: V(G) \to \{0,1,2\}$ as follows: set $f(y_i, w_1) = f(y_i, w_2) = 2$ for all $i \in [k]$. If $C_i \in C'$, set $f(c_i, w_1) = f(c_i, w_2) = 2$. If $C_i \in C \setminus C'$, set $f(c'_i, w_1) = f(d'_i, w_1) = f(c''_i, w_2) = f(d''_i, w_2) = 1$. Label the remaining vertices with 0. See Figure 2. Since C' is an exact cover, every $x_i \in X$ is in exactly one element of C'. So, (x_i, w_1) and (x_i, w_2) are satisfied, and it is simple to check that the rest of vertices of G are satisfied. Thus, f is a PIDF with $w(f) = 4(3q) + 4t = k$.

Conversely, assume that there exists a PIDF f on G such that $w(f) \leq k$. By Proposition 5 and Remark 2, $f(x_i, w_1) = f(x_i, w_2) = 0$ for every $i \in [3q]$; otherwise, we would have $w(f) > k$. By Proposition 6, $w\left(f\left(F_{x,k}\right)\right) \geq 4$. Observe that G contains $3q$ copies of $F_{x,k}$. By Proposition 7, $w\left(f(\mathrm{Q_c})\right) \geq 4$. Note that G contains t copies of Q_c. Since $w(f) \leq k$, we must have $w\left(f\left(F_{x,k}\right)\right) = 4$ for all the $3q$ copies of $F_{x,k}$, and $w\left(f(\mathrm{Q_c})\right) = 4$ for all the t copies of $\mathrm{Q_c}$. By Proposition 7, for every $i \in [t]$, either $f(c_i, w_1) = f(c_i, w_2) = 0$ or $f(c_i, w_1) = f(c_i, w_2) = 2$. Construct the subset $C' \subseteq C$ as follows: let $C_i \in C'$ if and only if $f(c_i, w_1) = 2$. By Proposition 6, for every $i \in [3q]$, (x_i, w_1) is out-satisfied with respect to $F_{x_i,k}$. So, for every $i \in [3q]$, (x_i, w_1) is adjacent to exactly one (c_j, w_1) with $f(c_j, w_1) = 2$. Thus, every $x_i \in X$ is in exactly one element in C'. Thus, C' is an exact cover. ■

Figure 2. The restriction of f on Q_{c_i}.

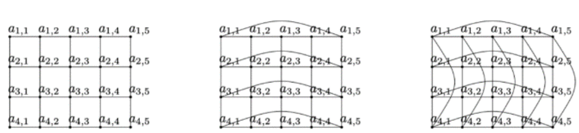


4. Bounds on $\gamma_I^p(G)$ of the Cartesian Product of Graphs

In this section, we show that if $G \in \{P_r \boxtimes P_s, C_r \boxtimes C_s\}$, where $r, s \geq 6$, $\gamma_I^p(G) \leq \frac{1}{2}|G| + \frac{1}{2}\min\{r, s\}$, and if $G = P_r \boxtimes C_s$, where $r \geq 6$, $\gamma_I^p(G) \leq \frac{1}{2}|G| + \frac{1}{2}r$. We also give an upper bound for $\gamma_I^p(G)$, where G is the Cartesian product of graphs. Then, we end this section with an open problem.

The graph $P_r \boxtimes P_s$ is a grid graph, $P_r \boxtimes C_s$ is a cylinder grid graph, and $C_r \boxtimes C_s$ is a torus grid graph with r rows and s columns. We denote the vertex in row i and column j by $a_{i,j}$; see Figure 3.

Figure 3. Cartesian products of paths and cycles.



Let $G \in \{P_r \boxtimes P_s, P_r \boxtimes C_s, C_r \boxtimes C_s\}$. Let $f: V(G) \to \{0,1,2\}$ be a function defined by

$$f\left(a_{i,j}\right) = \begin{cases} 1, & \text{if } j \text{ even}, \\ 1, & \text{if } j \in \{1, s\} \text{ and } i \equiv 1 \bmod 3, \\ 0, & \text{otherwise}. \end{cases}$$

To begin, assume that $G = P_r \boxtimes P_s$. We could presume that $r \leq s$.

Lemma 1. *If $r \equiv 0 \bmod 3$, $\gamma_I^p(G) \leq \frac{1}{2}|G| + \frac{1}{2}r$.*

Proof. Define a function $f': V(G) \to \{0,1,2\}$ by

$$f'\left(a_{i,j}\right) = \begin{cases} 1, & \text{if } j \in \{1, s\} \text{ and } i = r - 1, \\ f\left(a_{i,j}\right), & \text{otherwise}. \end{cases}$$

See Figure 4; we use red to highlight the vertices $a_{i,j}$ for which $f'\left(a_{i,j}\right) \neq f\left(a_{i,j}\right)$.

Clearly, f' is a PIDF on G. If s is odd,

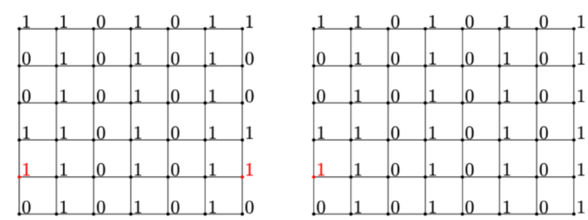
$$\begin{aligned} w(f') &= \frac{r(s-3)}{2} + r + \frac{2r}{3} + 2 \\ &= \frac{rs}{2} + \frac{r}{6} + 2 \\ &\underset{(1)}{\leq} \frac{|G|}{2} + \frac{r}{2}. \end{aligned}$$

Inequality (1) holds as $r \geq 6$. If s is even,

$$\begin{aligned} w(f') &= \frac{r(s-2)}{2} + r + \frac{r}{3} + 1 \\ &= \frac{rs}{2} + \frac{r}{3} + 1 \\ &\underset{(2)}{\leq} \frac{|G|}{2} + \frac{r}{2}. \end{aligned}$$

Inequality (2) holds as $r \geq 6$. Thus, the statement holds. ■

Figure 4. The function f' when $r \equiv 0 \; mod \; 3$.



Lemma 2. *If $r \equiv 1 \bmod 3$, $\gamma_I^p(G) \leq \frac{1}{2}|G| + \frac{1}{2}r$.*

Proof. It is not difficult to see that f is a PIDF on G.

If s is odd,

$$\begin{aligned} w(f) &= \frac{r}{2}(s-3) + r + \frac{2(r-1)}{3} + 2 \\ &= \frac{rs}{2} + \frac{r}{6} + \frac{4}{3} \\ &< \frac{rs}{2} + \frac{r}{2} \\ &= \frac{|G|}{2} + \frac{r}{2}. \end{aligned}$$

If s is even,

$$\begin{aligned} w(f) &= \frac{r(s-2)}{2} + r + \frac{r-1}{3} + 1 \\ &= \frac{rs}{2} + \frac{r}{3} + \frac{2}{3} \\ &< \frac{rs}{2} + \frac{r}{2}. \end{aligned}$$

Thus, the statement holds. ■

Remark 3. *Observe that if $G \in \{P_r \boxtimes C_s, C_r \boxtimes C_s\}$, $r \equiv 1 \bmod 3$ and s is odd, then f is a PIDF on G. From the above proof, $\gamma_I^p(G) < \frac{|G|}{2} + \frac{r}{2}$.*

Lemma 3. *If $r \equiv 2 \bmod 3$, $\gamma_I^p(G) \leq \frac{1}{2}|G| + \frac{1}{2}r$.*

Proof. Checking that f is a PIDF on G is simple, see Figure 5.

If s is odd,

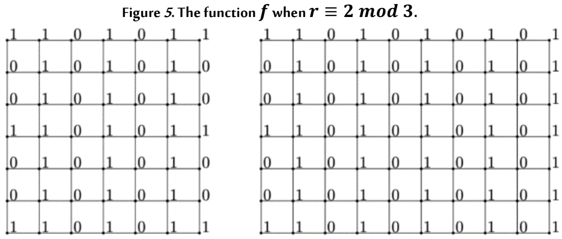
$$\begin{aligned} w(f) &= \frac{r(s-3)}{2} + r + \frac{2(r-2)}{3} + 2 \\ &= \frac{rs}{2} + \frac{r}{6} + \frac{2}{3} \\ &< \frac{rs}{2} + \frac{r}{2}. \end{aligned}$$

If s is even,

$$\begin{aligned} w(f) &= \frac{r(s-2)}{2} + r + \frac{r-2}{3} + 1 \\ &= \frac{rs}{2} + \frac{r}{3} + \frac{1}{3} \\ &< \frac{rs}{2} + \frac{r}{2}. \end{aligned}$$

So, the statements hold. ■

Figure 5. The function f when $r \equiv 2 \; mod \; 3$.



(a) s is odd (b) s is even

Remark 4. If $G = P_r \boxtimes C_s$, $r \equiv 2 \bmod 3$ and s is odd, then f is a PIDF on G. So, $\gamma_I^p(G) < \frac{|G|}{2} + \frac{r}{2}$.

Theorem 3. If $G = P_r \boxtimes P_s$, where $r, s \geq 6$, $\gamma_I^p(G) \leq \frac{1}{2}|G| + \frac{1}{2}\min\{r, s\}$.

Proof. Follows by Lemmas 1, 2 and 3. ■

Now, assume $G \in \{P_r \boxtimes C_s, C_r \boxtimes C_s\}$.

Lemma 4. If s is even, $\gamma_I^p(G) \leq \frac{1}{2}|G|$.

Proof. Let $g: V(G) \to \{0,1,2\}$ be a function defined by

$$g\left(a_{i,j}\right) = \begin{cases} 1, & \text{if } j \text{ is even,} \\ 0, & \text{if } j \text{ is odd.} \end{cases}$$

The function g is a PIDF on G with weight equal to $\frac{1}{2}|G|$. ■

Lemma 5. If s is odd and $r \equiv 0 \bmod 3$, $\gamma_I^p(G) \leq \frac{1}{2}|G| + \frac{1}{2}r$.

Proof. If $G = P_r \boxtimes C_s$, we slightly modify the function f by labelling $a_{i,j}$ with 1 if $j \in \{1, s\}$ and $i = r - 1$, keeping the rest of the labels without any change. This is the labelling used in Lemma 1 (the function f'), and we showed that the sum of all labellings is at most $\frac{1}{2}|G| + \frac{1}{2}r$.

If $G = C_r \boxtimes C_s$, then f is PIDF on G, and $w(f) < w(f')$. So, the statement holds. ■

Lemma 6. If s is odd and $r \equiv 2 \bmod 3$, $\gamma_I^p(G) \leq \frac{1}{2}|G| + \frac{1}{2}r$.

Proof. For $G = P_r \boxtimes C_s$, see Remark 4. For $G = C_r \boxtimes C_s$, define a function $f': V(G) \to \{0,1,2\}$ by

$$f'\left(a_{i,j}\right) = \begin{cases} 1, & \text{if } j \in \{1, s\} \text{ and } i = r, \\ f\left(a_{i,j}\right), & \text{otherwise.} \end{cases}$$

Checking that f' is a PIDF is simple, in addition,

$$\begin{aligned} w(f') &= \frac{r(s-3)}{2} + r + \frac{2(r-2)}{3} + 4 \\ &= \frac{rs}{2} + \frac{r}{6} + \frac{8}{3} \\ &\leq \frac{rs}{2} + \frac{r}{2}. \end{aligned}$$

The last inequality follows from the fact that $r \equiv 2 \bmod 3$ and $r \geq 6$. ■

Theorem 4. Let $r, s \geq 6$. If $G = P_r \boxtimes C_s$, $\gamma_I^p(G) \leq \frac{1}{2}|G| + \frac{1}{2}r$. If $G = C_r \boxtimes C_s$, $\gamma_I^p(G) \leq \frac{1}{2}|G| + \frac{1}{2}\min\{r, s\}$.

Proof. Follows by Lemmas 4–6, Remarks 3 and 4, and the symmetry between s and r when $G = C_r \boxtimes C_s$. ■

Looking at Theorems 3 and 4 and knowing that $\gamma_I^p(H) \leq \frac{1}{2}|H| + 1$ if H is a path or a cycle, it is natural to ask whether we can always drive an upper bound for $\gamma_I^p(G_1 \boxtimes G_2)$ in terms of $\gamma_I^p(G_1)$ or $\gamma_I^p(G_2)$. The following theorem answers this question.

Theorem 5. Let G_1 and G_2 be graphs and $G = G_1 \boxtimes G_2$. Assume $\gamma_I^p(G_1) = m_1$ and $\gamma_I^p(G_2) = m_2$. Then, $\gamma_I^p(G) \leq \min\{m_1|G_2|, m_2|G_1|\}$.

Proof. We can assume that $\min\{m_1|G_2|, m_2|G_1|\} = m_1|G_2|$. Let g be a PIDF witnessing that $\gamma_I^p(G_1) = m_1$. Define a function h on G as follows: for every $(v, u) \in G_1 \boxtimes G_2$, set $h(v, u) = g(v)$. Let (v, u) be any vertex in G. If $h(v, u) \neq 0$, it is done. So, assume $h(v, u) = 0$. Let $A := \{(v, u') \mid uu' \in E(G_2)\}$ and $B := \{(v', u) \mid vv' \in E(G_1)\}$; then, $N(v, u) = A \cup B$. Every vertex in A is labelled 0 as $h(v, u') = g(v) = h(v, u) = 0$. Since g is a PIDF on G_1, $\sum_{(v', u) \in B} h(v', u) = \sum_{v' \in N_{G_1}(v)} g(v') = 2$. Thus, h is PIDF on G and $w(h) \leq m_1|G_2|$. ■

Haynes and Henning (2019) showed that if T is a tree with $|T| \geq 3$, then $\gamma_I^p(T) \leq \frac{4}{5}|T|$. The following is a direct result of the previous theorem.

Corollary 1. If T_1 and T_2 are trees with $|T_1|, |T_2| \geq 3$, then $\gamma_I^p(T_1 \boxtimes T_2) \leq \frac{4}{5}|T_1 \boxtimes T_2|$.

Haynes and Henning (2019) proved that their result is tight. They showed that if n is a multiple of 5, there exists a tree T with $|T| = n$ such that $\gamma_I^p(T) = \frac{4}{5}n$. For example, let T' be the graph obtained from the star graph S_4 by adding an extra vertex adjacent to one of the leaves; then, $|T'| = 5$ and $\gamma_I^p(T') = \frac{4}{5}|T'|$. However, $\gamma_I^p(T' \boxtimes T') \leq 17$. We end this paper with the following open question.

Question. Is the bound in Corollary 1 tight?

5. Conclusions

In this paper, we proved that the perfect Roman domination and perfect Italian domination problems of the Cartesian product graphs are NP-complete. We also provided an upper bound for the perfect Italian domination number for the Cartesian product of a path and a path, a path and a cycle, a cycle and a cycle, and a tree and a tree. For future work, we suggest studying the complexity of the perfect Roman domination and perfect Italian domination of the Cartesian product of specific types of graphs. We also suggest finding the tight upper bound of the perfect Italian domination number of the Cartesian product of paths and cycles and investigating the perfect Roman and perfect Italian domination numbers of the Cartesian product of other types of graphs.

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