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The Zero-Divisor Graphs of Variation Monogenic Semigroups

الرسومات البيانيه لقواسم صفريه في شبة زمره دائرية منحرفة

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Abstract

The undirected graph $\Gamma(VS_{Mn})$ is the zero-divisor graph of the monogenic semigroup S_M with zero. The non-zero vertices x^i and x^j of this graph are adjacent whenever $i+j>n$ and $\gcd(i,j)=1$, where n is the order of $\Gamma(VS_{Mn})$. In this work, we consider some properties of the graph $\Gamma(VS_{Mn})$, such as the diameter, girth, chromatic number and clique. In addition, we show that $\Gamma(VS_{Mn})$ is a perfect, well-covered and coprime graph.

الملخص

الرسم البياني الغير الموجه $\Gamma(VS_{Mn})$ هو رسم بياني لشبه زمره دائرية S_M ذو قواسم صفريه مع محايد. الرؤوس الغير الصفرية x^i و x^j لهذا الرسم البياني يقال عنها أنها مرتبطة إذا حققت $i+j>n$ و gcd (i,j) =1 حيث أن العدد n يمثل عدد عناصر الرسم البياني $\Gamma(VS_{Mn})$. في هذا البحث تم دراسة خصائص هذا الرسم البياني $\Gamma(VS_{Mn})$ مثل القطر و مقاس المحيط و العدد اللوني و عدد العصب. وجد أن الرسم البياني $\Gamma(VS_{Mn})$ بيان كامل و مغطى جيدا و أولي نسبي.

1. Introduction

Zero-divisor graphs were first considered in commutative rings by Beck (1988). This subject was later applied to semigroups by DeMeyer *et al.* (2002), and it was then expanded by many researchers (for example, see DeMeyer *et al.* (2005) and Wright [2007]).

The literature on zero-divisor graphs of semigroups discusses many classifications of semigroups, such as commutative semigroups and monogenic semigroups. The zero-divisor graphs of monogenic semigroups were studied by Das *et al.* (2013). This particular work will continue this investigation.

The zero-divisor graph $\Gamma(S)$ for a commutative semigroup S with $\{0\}$ is an undirected graph whose vertices are the zero-divisor of S (DeMeyer *et al.*, 2002). In other words, the two vertices x and y in $Z(S)$ are adjacent when $xy=0$, where $Z(S)$ is the set of zero-divisors. In Das *et al.* (2013), the zero-divisor graph $\Gamma(S_M)$ for a monogenic semigroup S_M with $\{0\}$ is defined as an undirected graph whose nonzero vertices $x^i, x^j \in S_M$ are adjacent if they satisfy the following:

$x^i.x^j = 0$ only if $i+j > n$

where $1 \le i,j \le n$. In this work, we add one more condition to the graph obtained in Das *et al.* (2013) to obtain a new zero-divisors monogenic semigroup graph, and its characteristics will be studied. We will use the same notation as Das *et al.* (2013).

A *semigroup* is a set with associative binary operation. When a semigroup is generated by one element, it is called a monogenic semigroup. When S is a semigroup with $\{0\}$, the element $s \in S$ is called a *zero-divisor* if there is an element $t \in S$ that satisfies $s.t = 0 \Rightarrow s \neq 0$ or $t \neq 0$. Usually, the set of zero-divisors is denoted by $Z(S)$. Consider the monogenic semigroup S as such that $S_M = \{x, x^2, \dots, x^n\}$ with $\{0\}$ is defined as follows:

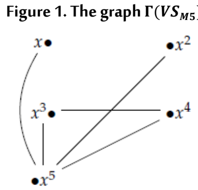
$x^i.x^j = x^{i+j} = 0$ only if $i+j>n$ and $\gcd(i,j)=1$. (1)

where x^i and $x^j \in S_M$ and $1 \le i,j \le n$. We will call this zero-divisors monogenic semigroup the *variation monogenic semigroup,* as denoted by VS_{Mn}, where n is the order of VS_{Mn}. VS_{Mn} is a semigroup. The undirected graph $\Gamma(VS_{Mn})$ on the n vertices is a graph whose nonzero zero-divisors vertices $x^i, x^j \in VS_{Mn}$ are only adjacent if the rule in (1) holds. If vertices x^i and x^j are adjacent, then we write $x^i x^j \in E(\Gamma(VS_{Mn}))$, and we call it the edge set of $\Gamma(VS_{Mn})$. The vertex set of $\Gamma(VS_{Mn})$ will be denoted by $V(\Gamma(VS_{Mn}))$. DeMeyer *et al.* (2002) denote the set of vertices, which are adjacent to x^i, by $ann(x^i)$.

1.1. Example:

Consider $\Gamma(VS_{M5})$, where $n=5$. The vertex set $V(\Gamma(VS_{M5})) = \{x, x^2, x^3, x^4, x^5\}$, and the graph is given in Figure 1 below:

Figure 1. The graph $\Gamma(VS_{M5})$

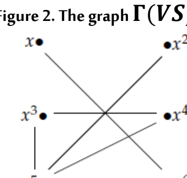


$ann(x^1) = \{x^5\}$, $ann(x^3) = \{x^4, x^5\}$, and $ann(x^5) = \{x^1, x^2, x^3, x^4\}$.

1.2. Example:

When considering $\Gamma(VS_{M6})$ given in Figure 2 below, the vertex set $V(\Gamma(VS_{M6})) = \{x, x^2, x^3, x^4, x^5, x^6\}$.

Figure 2. The graph $\Gamma(VS_{M6})$



$ann(x^1) = \{x^6\}$, $ann(x^3) = \{x^4, x^5\}$, and $ann(x^5) = \{x^2, x^3, x^4, x^6\}$.

The *diameter* of a graph G, as denoted by $diam$(G), is defined as follows:

$$diam(G) = max\{d_G(x,y): x, y \in V(G)\}.$$

where d_G is the distance between two vertices. By the distance, we mean the shortest path between two vertices. So, the diameter is the

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greatest distance between two vertices in the graph. However, the *radius* is the smallest distance between two vertices in the graph. The radius of a graph G, as denoted by $rad(G)$, is given as follows:

$$rad(G) = min\{d_G\ (x,y): x,y \in V(G)\}.$$

When there is no confusion about the graph in question G, we use $d(x,y)$ instead of $d_G(x,y)$ to represent the distance between x and y in G.

The *girth* of a graph is defined as the length of the shortest cycle in the graph. For any vertex x in G, the number of vertices that are adjacent to x is called the *degree* of the vertex x and is denoted by $deg_G(x)$. Then, $deg_G(x) = |ann(x)|$. The maximum degree for G is denoted by Δ(G), which refers to the largest vertex's degree in G. The minimum degree for Γ(G) is denoted by δ(G), which is the smallest vertex's degree in G. The non-increasing sequence of the vertices' degrees of graph G is called the *degree sequence* of G and denoted by $DS(G)$. The *irregularity index* of graph G is defined as the number of different terms in $DS(G)$ and denoted by $MWB(G)$, as in Das *et al.* (2013).

When considering the graph G and a subset D of $V(G)$, D is a *dominating set* for graph G if each vertex of $V(G)$ is adjacent to at least one vertex of D. The number of vertices of the smallest dominating set is called the *domination number* and denoted by γ(G).

The coloring of graph G is an assignment of colors to vertices of G so that no two adjacent vertices are assigned the same color. The minimum number of colors is called the *chromatic number* of G and is denoted by χ (G). A *coprime labelling* of a graph G of order n is a labeling of the vertices of G with distinct integers $1,2,\dots,n$, so that the labels on any two adjacent vertices are relatively prime. Any graph that admits coprime labelling is known as a *coprime graph*. If a subgraph of graph G is a complete graph, then this subgraph is called a *clique*. The number of vertices in the maximum clique (no other clique with more vertices) of G is called the *clique number* and denoted by ω(G). As stated by Lovasz (1972), graph G is a perfect graph if ω(G) $= \chi$ (G).

The *independent set* of graph G is a set of vertices in G that satisfies that no two vertices are adjacent in the set. The *independent number,* which is denoted by α(G), is the cardinality of a maximum independent set in G. A graph G is said to be a *well-covered* graph when all its maximal independent sets are maximum.

In the next section, we give the characteristics of a variation monogenic semigroup.

2. Results

In this section, some properties of the graph $\Gamma(VS_{Mn})$ are investigated, such as girth, diameter, maximum degree, minimum degree, degree sequence, domination number, clique number, chromatic number, independent set, well-covered and coprime graph.

The next result gives the diameter of $\Gamma(VS_{Mn})$.

2.1. Theorem:

When VS_{Mn} is a variation monogenic semigroup, the diameter of $\Gamma(VS_{Mn})$ is as follows:

$$diam\Gamma(VS_{Mn}) = \begin{cases} 1, n = 2 \\ 2, n > 2\ and\ prime \\ 3, if\ otherwise \end{cases}$$

Proof: First, when we consider the graph $\Gamma(VS_{Mn})$ with $n = 2$, It is obvious that the greatest path in $\Gamma(VS_{Mn})$ is 1, which provides the diameter of the graph. Next, we split the remaining problem into the following two cases:

Case 1: when n is a prime number. The vertex x^n is adjacent to all the vertices in $\Gamma(VS_{Mn})$ since the gcd $(i,j) = 1$ and $n + i > n$, for every vertex x^i in $\Gamma(VS_{Mn})$, where $i < n$. Hence, the result follows from Theorem 1 in Das *et al.* (2013).

Case 2: when n is not a prime number. When there are the vertices x^i, x^j, x^r in $\Gamma(VS_{Mn})$ so that x^i and x^j are not adjacent, gcd $(n,j) \neq 1$ and gcd $(i,n) =$ gcd $(r,n) =$ gcd $(j,r) = 1$, where $1 < j, r < n$ and $1 \leq i < n$. Vertex x^i is adjacent to only vertex x^n or x^{n-1} since $i + n > n$ and gcd $(i,n) = 1$. The diamater of $\Gamma(VS_{Mn})$ can be viewed as the distance between vertex x^i and vertex x^j. This is $x^i - x^n - x^r - x^j$, which gives $diam(\Gamma(VS_{Mn})) = 3$, as required.

2.2. Theorem:

Let VS_{Mn} be a variation monogenic semigroup. Then, $rad(\Gamma(VS_{Mn})) = 1$.

The proof is straightforward since the vertex x is adjacent to vertex x^n only.

2.3. Theorem:

When VS_{Mn} is a variation monogenic semigroup, the girth of $\Gamma(VS_{Mn})$ is 3 for $n > 4$.

Proof: Let p denote the highest prime number that is less or equal to n. The definition of $\Gamma(VS_{Mn})$ shows that $x^p x^{p-1} = 0$ and $x^{p-1} x^{p-2} = 0$. Additionally, we have $x^p x^{p-2} = 0$ for $2(p-1) > n$. Therefore, we have $x^p - x^{p-1} - x^{p-2} - x^p$. Therefore, the girth is 3 for $n > 4$.

For $n \leq 4$, $x^{p-1} x^{p-2} \neq 0$ since $2p - 3$ is not greater than n. Therefore, $x^{p-1}x^{p-2} \notin E(\Gamma(VS_{Mn}))$,which implies that there is no cycle $x^p - x^{p-1} - x^{p-2} - x^p$ for $n \leq 4$. Hence, the girth does not exist for $n \leq 4$.

2.4. Theorem:

When p is the highest prime number that is less or equal to n, then for any variation monogenic semigroup VS_{Mn}, the maximum degree and minimum degree of $\Gamma(VS_{Mn})$ are $\Delta(\Gamma(VS_{Mn})) = p - 1$ and $\delta(\Gamma(VS_{Mn})) = 1$.

Proof: For $p = n$, the result follows from Das *et al.* (2013). When we assume that $p \neq n$, vertex x^p has the maximum degree. Therefore, $x^p x^i = 0$ if $p + i > n$ and gcd $(p,i) = 1$. This implies that $x^p x^i = 0$ for all $i > n - p$, that is $n - p < i \leq n$. Furthermore, deg $(x^p) = |i|$, where $|i| = (n - (n - p)) - 1 = p - 1$ $(x^p x^p \notin E(\Gamma(VS_{Mn}))$ since $\Gamma(VS_{Mn})$ is a simple graph). Hence, the maximum degree is $p - 1$.

Additionally, by definition of $\Gamma(VS_{Mn})$, x is adjacent to only x^n. Therefore, the minimum degree is 1.

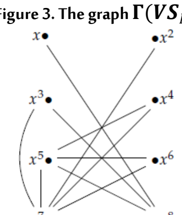
2.5. Corollary:

$|ann(x^p)| = p - 1$, where p is the highest prime number that is less or equal to n.

2.6. Example:

Consider the graph $\Gamma(VS_{M8})$ with the vertex set $\{x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\}$ given in Figure 3.

Figure 3. The graph $\Gamma(VS_{M8})$



This graph has a $diam(\Gamma(VS_{M8})) = 3$, $rad(\Gamma(VS_{M8})) = 1$, girth$\big(\Gamma(VS_{M8})\big) = 3$,

$\Delta\ (\Gamma(VS_{M8})) = 6$ and $\delta(\Gamma(VS_{M8})) = 1$.

2.7. Theorem:

For any positive integer of n, i and r with $1 \le i \le n$, let $K_i = \{r: n - i < r \le n$ and $\gcd\,(i, r) = 1\}$ and $m_i = |K_i|$, then the degree sequence of the graph $\Gamma(VS_{Mn}) = m_1, m_2, \dots, m_n$.

The proof follows directly from the definition of $\Gamma(VS_{Mn})$. Below is an example that illustrates the result.

2.8. Example:

Consider the graph $\Gamma\ (\ VS_{M6}\ )$ with the vertex set $\{x, x^2, x^3, x^4, x^5, x^6\}$. Let K_i be the degree of vertex x^i, where $x^i \in V(\Gamma(VS_{M6})$. Then, $|K_1| = |\{6\}| = 1$, $|K_2| = |\{5\}| = 1$, $|K_3| = |\{4, 5\}| = 2$, $|K_4| = |\{3, 5\}| = 2$, $|K_5| = |\{2, 3, 4, 6\}| = 4$, and $|K_6| = |\{1, 5\}| = 2$. Therefore, the degree sequence of $\Gamma(VS_{M6}) = 1, 1, 2, 2, 2, 4$.

2.9. Remark:

Let C* denote the set of prime numbers that are less or equal to n. The irregularity index of $\Gamma(VS_{Mn})$ is given as |C*|.

2.10. Definition:

Let p be the greatest prime number that is less than n, and let C denote the class of x^i so that $n + i > n$ and $\gcd\,(i, n) \neq 1$ for $1 \le i \le n - p$. Additionally, let Q denote the class of vertices x^{ik} so that $x^k \notin$ C where k is prime and $1 \le k \le n - p$.

Note that if C is empty then Q is empty.

2.11. Theorem:

For the positive integer n,

$$\gamma(\Gamma(VS_{Mn}) = \begin{cases} 1, n\ is\ prime \\ 2, C, Q\ are\ empty \\ 3, C\ is\ not\ empty\ and\ Q\ is\ empty \\ 4, C, Q\ are\ empty \end{cases}$$

Proof: We split the problem into the following four cases:

Case 1: When n is prime,

n is relatively prime to i, where $i = 1, 2, \dots, n-1$. When $D \subset V\big(\Gamma(VS_{Mn})\big)$, it follows from the definition of $\Gamma(VS_{Mn})$ that $x^n x^i \in E\ (\Gamma(VS_{Mn}))$ for all i. Hence, $D = \{x^n\}$ is the dominating set of $\Gamma(VS_{Mn})$ with cardinality one whenever n is prime.

Case 2: When C and Q are empty, it follows from the definition of C that there is i for $1 \le i \le n - p$ so that $\gcd\,(i, n) = 1$ since C is empty. When $D = \{x^n, x^p\} \subset V(\Gamma(VS_{Mn}))$, we have $x^n x^i \in E\ (\Gamma(VS_{Mn}))$ for all x^i, $1 \le i \le n - p$, which follows from the definition of $\Gamma(VS_{Mn})$. Additionally, since p is the greatest prime number that is less or equal to n, we have $x^p x^t \in E\ (\Gamma(VS_{Mn}))$ for all x^t, where $n - p + 1 \le t \le n$. Therefore, $D = \{x^n, x^p\}$ is the dominating set with cardinality 2.

Case 3: When C is not empty and Q is empty. Since C is not empty, it follows that there is $1 \le i, j \le n - p$ so that $\gcd\,(n, i) = 1$ and $\gcd\,(n, j) \neq 1$. Also, there is a vertex x^r so that $\gcd\,(r, j) = 1$ and $p < r < n$. Let $D = \{x^n, x^r, x^p\} \subset V(\Gamma(VS_{Mn}))$ and $r + j > n$. Then, it is sufficient to show that D is the dominating set for $\Gamma(VS_{Mn})$. Consider the set of vertices $\{x^t\}$ so that $n - p + 1 \le t \le n$. The set of vertices x^t is adjacent to vertex x^p since p is prime and $t + p > n$. Next, consider the set of vertices $x^i \notin C$, where $1 \le i \le n - p$. It follows that x^n is adjacent to all x^i since $\gcd\,(n, i) = 1$ and $n + i > n$. Lastly, consider the set of vertices $\{x^j\} \in C$, where $1 \le j \le n - p$. Vertex x^r is adjacent to all $\{x^j\}$ since $\gcd\,(r, j) = 1$ and $r + j > n$. Hence, $D = \{x^n, x^r, x^p\}$ is the domnating set with cardinality 3.

Case 4: When C and D are not empty. Let $D = D' \cup D''$ where D' is the dominating set of $\Gamma(VS_{Mn})$ when C is not empty and D'' is the dominating set for the class Q. It follows from case 3 above that $|D'| = 3$. Let $1 \le i, k, q \le n - p$ so that $\gcd\,(n, i) \neq 1$ and $\gcd(n, q) \neq 1$. Note that $q = ki$ where k is a prime number with $x^k \notin C$. Next, consider the class Q containg the set of vertices $x^q = x^{ki}$. Since $\gcd\,(n, i) \neq 1$, it follows from the definition of $\Gamma(VS_{Mn})$ that $\gcd\,(n, q) \neq 1$, which implies that $x^n x^q \notin E(\Gamma(VS_{Mn}))$. Also, $x^q x^p \notin E(\Gamma(VS_{Mn}))$ since $p + q \le n$. By case 3 above and from the definition of class C, we have $\gcd\,(r, i) = 1$ and $\gcd(r, k) \neq 1$ for $p + 1 \le r \le n$, which implies that $\gcd\ (r, q) \neq 1$ and so $x^r x^q \notin E(\Gamma(VS_{Mn}))$. Therefore, there is $\{x^s\}$ for $p < s \le n$ so that $D'' = \{x^s\}$, where $s + q > n$ and $\gcd(s, q) = 1$. Hence, $D = D' \cup D'' = \{x^n, x^r, x^p\} \cup \{x^s\}$ is the dominating set with cardinality 4.

2.12. Example:

Consider the graphs $\Gamma(VS_{M9})$ and $\Gamma(VS_{M10})$. The class C and Q are empty in the case of $\Gamma(VS_{M9})$. Additionally, class C is not empty and Q is empty in the case of $\Gamma(VS_{M10})$. Therefore, $\gamma\ (\Gamma(VS_{M9})) = |\{x^7, x^9\}| = 2$ and $\gamma\ (\Gamma(VS_{M10})) = |\{x^7, x^9, x^{10}\}| = 3$.

2.13. Definition:

For the positive integer n, let $\pi(n) = \{p_1, p_2, \dots, p_k\}$ be the set of consecutive prime numbers that is less or equal to n that satisfies $p_t + p_{t+1} > n$. The total number of elements in $\pi(n)$ is denoted by $\pi^*(n) = |\pi(n)|$.

2.14. Definition:

Let $\pi(n)$ be given as in Definition 2.12. Define s_f^* as the positive integers $\left\lceil\frac{n}{2}\right\rceil < s_f^* \le n$ so that the following happens:

1) $s_f^* \notin \pi\ (n)$

2) $\gcd\ (s_f^*, p_f) = 1$ for all $p_f \in \pi\ (n)$

3) $\gcd\ (s_f^*, s_l^*) = 1$ for any pair $\left\lceil\frac{n}{2}\right\rceil < s_f^*, s_l^* \le n$

The number of s_f^* is defined as $S^* = |s_f^*|$

2.15. Theorem:

Let VS_{Mn} be a variation monogenic semigroup. The chromatic number of $\Gamma(VS_{Mn})$ is given as $\pi^*(n) + S^*$.

Proof: Note that $x^{p_f} x^{p_l} \in$ E$(\Gamma(VS_{Mn}))$ for any $p_f, p_l \in \pi\ (n)$. Therefore, we need $\pi^*(n)$ distinct colors to color all the vertices x^{p_f} where $p_f \in \pi\ (n)$. Let color C_1 be assigned to vertex x^{p_1}, where $p_1 \in \pi(n)$, then it is obvious that $x^r x^{p_1} \notin$ E $(\Gamma(VS_{Mn}))$ for r prime and $r \notin \pi\ (n)$. This implies that color C_1 can be assigned to all vertices x^r.

Next, consider the set of vertices $x^{s_f^*}$ so that $x^{s_f^*} \notin \pi\ (n)$, $\left\lceil\frac{n}{2}\right\rceil < s_f^* \le$ n. For any pair $s_f^*, s_l^*, x^{s_f^*} x^{s_l^*} \in$ E $(\Gamma(VS_{Mn}))$ and $x^{p_f} x^{s_f^*} \in$ E $(\Gamma(VS_{Mn}))$, which follows from the definition of s_f^* and $\Gamma(VS_{Mn})$. There is a need of an additional S* colors to color the set of vertices $\{x^{s_f^*}\}$. Furthermore, the set of vertices $\{x^m\}$, where $1 < m \le n$, can be assigned the same color that was assigned to vertex $\{x^{s_f^*}\}$ if $\gcd\ (m, s_f^*) \neq 1$.

Lastly, vertex x can be assigned any color that was assigned to the vertices of the graph, provided that the color is different from the color that was assigned to vertex x^n. Hence, the total number of colors needed to color the graph $\Gamma(VS_{Mn})$ is $\pi^*(n) + S^*$.

2.16. Theorem:

When VS_{Mn} is a variation monogenic semigroup, the clique number of $\Gamma(VS_{Mn})$ is given as $\pi^*(n) + S^*$.

Proof: Note that all the vertices $x^{p_i}, p_i \in \pi(n)$ are adjacent to each other, which follows from the definition of $\pi(n)$ and $\Gamma(VS_{Mn})$. Next, consider the set of vertices $\{x^{s_f^*}\}$ where $\left\lceil\frac{n}{2}\right\rceil < s_f^* \leq n$. It is clear that $x^{p_i}x^{s_f^*} \in E(\Gamma(VS_{Mn}))$ since $gcd\ (p_i, s_f^*) = 1$. In addition, let $\{x^t\}$ be the set of vertices in $\Gamma(VS_{Mn})$ so that $1 \leq t < p_1$ and $p_1 \in \pi(n)$. It is obvious that $x^t x^{p_1} \notin E(\Gamma(VS_{Mn}))$ since $t + p_1 < n$. Also, for the pairs of vertices $x^{s_f^*}, x^{s_l^*}$; $x^{s_f^*}x^{s_l^*} \in E(\Gamma(VS_{Mn}))$ since $\gcd\ (s_f^*, s_l^*) = 1$ and $s_f^* + s_l^* > n$, which follows from the definition s_f^*. Hence, the clique number of $\Gamma(VS_{Mn})$ is given as $\pi^*(n) + S^*$.

2.17. Remark:

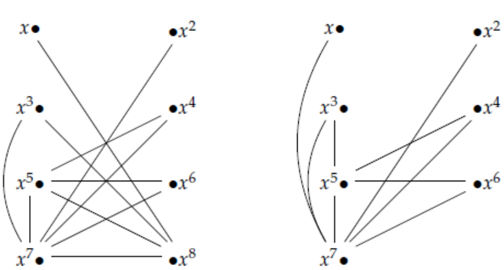
The graph $\Gamma(VS_{Mn})$ is a perfect graph since $\chi\ (\Gamma(VS_{Mn})) = \omega\ (\Gamma(VS_{Mn})) = \pi^*(n) + S^*$.

2.18. Example:

For the graph $\Gamma(VS_{M7})$, we have $\pi(7) = \{3, 5, 7\}$ and so $\pi^*(7) = 3$. Also, $S^* = 0$. Then, $\chi(\Gamma(VS_{M7})) = \pi^*(7) + S^* = 3 + 0 = 3 = \omega\ (\Gamma(VS_{M7}))$.

For the graph $\Gamma(VS_{M8})$, we have $\pi(8) = \{5, 7\}$ and so $\pi^*(8) = 2$. Moreover, $S^* = 1$. Then, $\chi(\Gamma(VS_{M8})) = \pi^*(8) + S^* = 2 + 1 = 3 = \omega\ (\Gamma(VS_{M8}))$.

Figure 4. The graph $\Gamma(VS_{M8})$ and $\Gamma(VS_{M7})$



For $\Gamma(VS_{M7})$, the minimum number of colors is three, and they are $C_1 = \{x^7\}$, $C_2 = \{x^5, x^2, x\}$ and $C_3 = \{x^6, x^4, x^3\}$. The complete subgraphs (*cliques*) from $\Gamma(VS_{M7})$ are $\{x^5, x^6, x^7\}$. Hence, $\omega(\Gamma(VS_{M7})) = 3$.

For $\Gamma\ (VS_{M8})$, the minimum number of colors is three, which are $C_1 = \{x^8, x^6, x^4, x^2\}$, $C_2 = \{x^7, x\}$ and $C_3 = \{x^5, x^3\}$. The complete subgraphs (*cliques*) from $\Gamma(VS_{M8})$ are $\{x^5, x^6, x^7\}$. Hence, $\omega(\Gamma(VS_{M8})) = 3$.

2.19. Lemma:

Let VS_{Mn} be a variation monogenic semigroup, then $\alpha(\Gamma(VS_{Mn})) \geq \left\lceil\frac{n}{2}\right\rceil$.

Proof: We know that $\frac{n}{2} \leq \left\lceil\frac{n}{2}\right\rceil$. For all $1 \leq i, j \leq \left\lceil\frac{n}{2}\right\rceil$, it is clear that $i + j \leq n$. Hence, x^i is not adjacent with x^j. Therefore, $\alpha(\Gamma(VS_{Mn})) \geq \left\lceil\frac{n}{2}\right\rceil$.

2.20. Remark

Let n be a positive integer. Define z_f^* as the positive integers where $\left\lceil\frac{n}{2}\right\rceil < z_f^* < n$ so that the following happens:

1) z_f^* is not prime

2) $\gcd\ (z_f^*, h_v) \neq 1$, or when $\gcd\ (z_f^*, h_v) = 1$, then $z_f^* + h_v \leq$ n for all $1 < h_v \leq \left\lceil\frac{n}{2}\right\rceil$

The number of z_f^* is defined as $Z^* = |\,z_f^*\,|$. Hence, $\alpha(\Gamma(VS_{Mn})) = \left\lceil\frac{n}{2}\right\rceil + Z^*$.

2.21. Theorem:

The graph $\Gamma(VS_{Mn})$ is a well-covered graph.

Proof: We need to show that all maximal independent sets $\alpha(\Gamma(VS_{Mn}))$ given in Remark 2.19 are maximum.

Next, for z_f^* and h_v, as defined in Remark 2.19, suppose there is a vertex $x^{c^*} \neq x^{z_f^*}$, where $x^{c^*} \in V(\Gamma\ (VS_{Mn}))$ and $c^* > \left\lceil\frac{n}{2}\right\rceil$. Additionally, let x^{h_v}, $x^{c^*} \in \alpha^*(\Gamma(VS_{Mn}))$ for some maximal independent sets $\alpha^*(\Gamma(VS_{Mn}))$ in $\Gamma(VS_{Mn})$. Note that $\gcd\ (c^*, h_v) = 1$ and $c^* + h_v > n$ since $x^{c^*} \neq x^{z_f^*}$, which follows on from Remark 2.19. This implies that $x^{c^*}x^{h_v} \in E(\Gamma(VS_{Mn}))$, which contradicts the assumption that $\alpha^*(\Gamma(VS_{Mn}))$ is a maximal independent set. Therefore, $x^{c^*} = x^{z_f^*}$, which means that every maximal independent set is maximum in $\alpha(\Gamma(VS_{Mn}))$. Hence, $\alpha(\Gamma(VS_{Mn}))$ is a well-covered graph.

2.22. Proposition:

Let $\pi(n) = \{p_1, p_2, \dots, p_k\}$ be the set of consecutive prime numbers that are less or equal to n so that $p_t + p_{t+1} > n$. In graph $\Gamma\ (VS_{Mn})$, we have the following:

i) If $i, j \in \pi(n)$, then there is a cycle of length 3 that contains x^i and x^j.

ii) If $i, j \notin \pi(n)$, and $i, j \neq 1, 2$ are prime numbers, then there is a cycle of length 4 that contains x^i and x^j.

Proof: i) Since $i, j \in \pi(n)$, then $i + j > n$ and $\gcd\ (i, j) = 1$. Hence, x^i is adjacent to x^j. Now there are two cases as follows:

a) If $i < j$, take $k = n - i + \delta$ where $0 \leq \delta \leq n$. δ guarantees that $k \neq i, j$. Since j is prime and $k \neq j$, then $\gcd\ (k, j) = 1$. Also, $k + j = n - i + \delta + j > n$, since $i < j$. Therefore, x^j is adjacent to x^k. Moreover, since i is prime and $k \neq i$, then $\gcd\ (k, i) = 1$. Also, $k + i = n - i + \delta + i = n + \delta > n$. Then, x^k is adjacent to x^i. Therefore, $x^i - x^j - x^k - x^i$ is a cycle of length 3, which contains x^i and x^j.

b) If $j < i$. Take $k = n - j + \delta'$, where $0 \leq \delta' \leq n$. δ' guarantees that $k \neq i, j$. The result uses the same arguments in case (a) above.

ii) Since i, j are prime numbers and neither of them are in $\pi(n)$, then $i + j \leq n$. Hence, x^i is not adjacent to x^j. This means that it is impossible to find a cycle of length 3 that contains x^i and x^j. Also, $i, j < n$ and $i, j < n - 1$ since i, j are not in $\pi(n)$. Then, since $i, j \neq 1, 2$ we have that $i + n > n, i + (n - 1) > n, j + n > n, j + (n - 1) > n$. Therefore, $x^i - x^n - x^j - x^{n-1} - x^i$ is a cycle of length 4 that contains x^i and x^j.

2.23. Theorem:

Let $x^i - x^j - x^z$ be a path of length 3 in $\Gamma\ (VS_{Mn})$. If $i = 1$ or 2, then $|ann(x^i) \cap ann\ (x^z)| = 1$.

Proof: We have split the problem into the following two cases:

Case 1: When $i = 1$. Recall that $ann\ (x^1) = x^n$ and $xx^n \in E(\Gamma(VS_{Mn}))$, which follows from the definition of $\Gamma(VS_{Mn})$. Therefore, we have $j = n$, which implies that $x^n x^z \in E\ (\Gamma(VS_{Mn}))$, then $x^n \in ann\ (x^z)$. Therefore, $x^n \in ann\ (x^i) \cap ann\ (x^z)$. Hence, $|ann(x^i) \cap ann(x^z)| = 1$ since $|ann(x^i)| = \{x^n\} = 1$.

Case 2: When $i = 2$. If n is odd, then x^2 is adjacent to x^n and not adjacent to x^n. Hence, the result follows from case 1 above. If n is even, then $x^2 x^{n-1} \in E(\Gamma(VS_{Mn}))$, which follows from the definition of $\Gamma(VS_{Mn})$. Therefore, $x^2 x^{n-1} \in E(\Gamma(VS_{Mn}))$, which implies that $j = n - 1$. Also, for $x^j x^z \in E(\Gamma\ (VS_{Mn}))$, $j + z > n$ and $\gcd\ (j, z) = 1$. This is true since $n - 1 + z > n$ for $z > 1$. Therefore, $x^{n-1} \in ann(x^z)$. That is, $x^{n-1} \in ann(x^i) \cap ann\ (x^z)$. Hence, $|ann(x^i) \cap ann\ (x^z)| = 1$ since $|ann(x^i)| = \{x^{n-1}\} = 1$.

From Proposition 2.21 and Theorem 2.22, the following can be deduced:

2.24. Corollary:

A cycle in $\Gamma(VS_{Mn})$ cannot contain vertex x^i, where $i = 1\ or\ 2$.

2.25. Corollary:

In $\Gamma(VS_{Mn})$, we have the following:

1) $ann(x^1) = x^n$,

2) $ann(x^2) = \begin{cases} x^n, n \text{ is odd} \\ x^{n-1}, n \text{ is even} \end{cases}$

2.25. Proposition:

A pentagon $x^i - x^j - x^k - x^f - x^t - x^i$ cannot be in $\Gamma(VS_{Mn})$ whenever $n < 8$.

Proof: We will prove this using a contradiction argument. Suppose there is a pentagon $x^i - x^j - x^k - x^f - x^t - x^i$ when $n < 8$. From Corollary 2.24, we get $i, j, k, f, t \neq 1 \text{ or } 2$. Now we have the following two cases:

Case 1: When $n = 7$, since $i, j, k, f, t \neq 1 \text{ or } 2$, then $3 \leq i, j, k, f, t \leq 7$. The vertex x^6 is not adjacent to x^3 or x^4. Hence, x^6 is adjacent to x^5 and x^7. Now there are two vertices left, which are x^3 and x^4. However, x^3 is not adjacent to x^4. Therefore, it is impossible to find a pentagon $x^i - x^j - x^k - x^f - x^t - x^i$.

Case 2: When $n < 7$, since $i, j, k, f, t \neq 1 \text{ or } 2$, then there are only four vertices or less. However, to create a pentagon, we need five different vertices.

Lastly, we have the following result:

2.26. Theorem:

The graph $\Gamma(VS_{Mn})$ is a coprime graph.

Proof: Recall from definition of $\Gamma(VS_{Mn})$ that for any pair of vertices, x^i, x^j; $x^i x^j \in E(\Gamma(VS_{Mn}))$ only if $i + j > n$ and $\gcd(i, j) = 1$. Combining definitions of $\Gamma(VS_{Mn})$ and coprime labeling gives the desired result.

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