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Controllability of Nonlocal Impulsive Semilinear Differential Inclusions with Fractional Sectorial Operators and Infinite Delay

Feryal Abdullah Al Adsani and Ahmed Gamal Ibrahim

Department of Mathematics, College of Science, King Faisal University, AlAsa, Saudi Arabia

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ABSTRACT

This paper demonstrates the controllability of two fractional nonlocal impulsive semilinear differential inclusions with infinite delay, where the linear part is a fractional sectorial operator and the nonlinear term is a multivalued function. The operator families generated by the linear part are not assumed to be compact. The objective is achieved using the properties of fractional sectorial operators and the Hausdorff measure of noncompactness. The results generalise several recent findings, and the method can be used to extend further contributions to cases where the linear term is a fractional sectorial operator and the nonlinear term is a multivalued function, in the presence of instantaneous impulses and infinite delays. The novelty of this work lies in initiating the study of the controllability of a system involving a fractional Caputo derivative under infinite impulses and delays. An example is presented to verify the theoretical developments. Given the wide-ranging applications of fractional calculus in medicine, energy and other scientific fields, this work contributes to those domains.

KEYWORDS

Caputo derivative, mazur's lemma, mild solutions, multivalued functions, noncompact measure, phase space

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1. Introduction

Fractional calculus has many applications in industry, energy, fluid flow, control theory, electrical circuits, electrochemistry, engineering, polymer science, organic dielectric materials, viscoelastic materials, rheology, diffusive transport, electrical networks, electromagnetic theory and physics (Baleanu and Lopes, 2019; Butt *et al.*, 2023; Sudsutad *et al.*, 2024). Many phenomena in real life are characterised by sudden changes in state and are modelled by impulsive differential equations and impulsive differential inclusions. For example, consider the motion of an elastic ball bouncing vertically on a surface. The instants of impulses occur when the ball meets the surface and its velocity changes rapidly. One of the recent works on this topic was presented by Aladsani and Ibrahim (2024).

Sectorial operators have many applications in partial differential equations. In Wang *et al.* (2015), the existence of mild solutions for fractional differential inclusions with fractional sectorial operators, impulsive effects and nonlocal conditions was demonstrated. It is known that nonlocal Cauchy problems are motivated by physical problems and various phenomena, such as nonlocal neural networks and nonlocal pollution. Many findings on nonlocal differential equations and inclusions have been reported (Hassan *et al.*, 2022; Zhang *et al.*, 2019).

A system is said to be controllable if a control function exists that directs the solution of the system from its initial state to its final state. This subject is of interest because many control processes can be represented as differential equations or inclusions. Approximate controllability means that the system can be steered to an arbitrarily small neighbourhood of the desired final state. In recent years, several achievements have been made regarding exact controllability (Almarri and Elshenhab, 2022; Alsaroria and Ghadle, 2022; Alsheekhhussain and Ibrahim, 2021) and approximate controllability (Varun *et al.*, 2022; Dineshkumar and Udhayakumar, 2022; Dineshkumar *et al.*, 2022; Kumar, 2023).

Wang *et al.* (2020) studied the finite controllability of Hilfer fractional semilinear differential equations, while Chalishajar *et al.* (2024) discussed the null controllability of Hilfer fractional stochastic differential equations with nonlocal conditions. Some authors have considered the controllability of problems with finite delay (Almarri and Elshenhab, 2022; Karthikeyan *et al.*, 2021), infinite delay (Bose and Udhayakumar, 2023; Slama and Boudaoui, 2017) and impulses (Wang *et al.*, 2019).

Different kinds of controllability have been investigated when the linear part is the infinitesimal generator of a semigroup of operators (Wang *et al.*, 2019), a fractional sectorial operator (Alsaroria and Ghadle, 2022; Raja *et al.*, 2022), or a fractional almost sector operator (Varun *et al.*, 2022).

Furthermore, some authors have treated problems involving integer-order systems (Karthikeyan *et al.*, 2021), and others have addressed systems with the fractional Caputo derivative (Almarri and Elshenhab, 2022; Wang *et al.*, 2019), the Atangana–Baleanu derivative (Dineshkumar *et al.*, 2022), the Riemann–Liouville derivative (Yang and Wang, 2016) and the Hilfer–Katugampola derivative (Hassan *et al.*, 2022). For contributions concerning the controllability of problems with nonlocal conditions, see Kumar *et al.* (2020) and Slama and Boudaoui (2017).

Nonetheless, the number of works in the literature on the controllability of problems involving infinite state-dependent delay and impulsive effects is limited, particularly when the right-hand side is a multi-valued function. To the best of current knowledge, the controllability of Caputo fractional differential inclusions generated by fractional sectorial operators in the presence of nonlocal conditions, impulsive effects and infinite delay has not yet been addressed.

The objective of this article is to examine the existence of mild solutions and to study the exact controllability of two impulsive fractional differential inclusions with fractional sectorial operators in the presence of infinite delay and nonlocal conditions, with the following structures:

Corresponding Author: Feryal Abdullah Al Adsani 0096666505934144, faladsani@kfu.edu.sa

$$(1)\quad \begin{cases} {}^cD^{\gamma}w(\vartheta) \in Aw(\vartheta) + \mathcal{F}(\vartheta, w_\vartheta) + Q(\mho(\vartheta)), a.e. on\ T - \{\vartheta_1, \vartheta_2, \dots, \vartheta_m\}, \\ w(\vartheta_k^+) = w(\vartheta_k^-) + I_k(w(\vartheta_k^-)), k = 1,2, \dots, m, \\ w(\vartheta) = \Psi(\vartheta) - g(w)(\vartheta), \vartheta \in (-\infty, 0]. \end{cases}$$

and

$$(2)\quad \begin{cases} {}^cD^{\gamma}w(\vartheta) \in Aw(\vartheta) + \mathcal{F}(\vartheta, w_\vartheta) + (Vz)(\vartheta), a.e. on\ T - \{\vartheta_1, \vartheta_2, \dots, \vartheta_m\}, \\ w(\vartheta_k^+) = w(\vartheta_k^-) + I_k(w(\vartheta_k^-)), k = 1,2, \dots, m, \\ w(\vartheta) = \Psi(\vartheta) - g(w)(\vartheta), \vartheta \in (-\infty, 0], \end{cases}$$

where $m \geq 2$ is a natural number, $\gamma \in (0,1), T = [0, \varpi], \varpi > 0$, ${}^cD^{\gamma}w(\vartheta)$ is the Caputo derivative of order γ (Kilbas *et al.*, 2006), E is a real Banach space with $dim(E) = \infty, \omega: T \to E, A: D(A) \subseteq E \to E$ is a fractional sectorial operator as in Wang *et al.* (2015), and the function $w_\vartheta(s) = w(\vartheta + s), s \in (-\infty, 0]$, belongs to some abstract phase space $\beta, \mathcal{F}: T \times \beta \to 2^E$ is a multi-valued function with non-empty values, $0 = \vartheta_0 < \vartheta_1 < \cdots < \vartheta_m < \vartheta_{m+1} = \varpi$, $I_k: E \to E\ (k = 1,2, \dots, m)$ are impulsive functions characterising the jump of the solutions at impulse points, $\Psi \in \beta$ and $g: \beta_\varpi \to \beta$ is a nonlinear function related to the nonlocal condition, and β_ϖ will be defined later.

In Problem (1), the control function $\mho \in L^\infty(T, X)$, where X is a real Banach space and $Q: X \longrightarrow E$ is a bounded linear operator. In Problem (2), the control function $z \in L^p(T, X), p > \frac{1}{\gamma}$ and $V: L^p(T, X) \to L^p(T, E)$ is a bounded linear operator. The spaces β_ϖ and $PC(T, E)$ will be defined later.

It is worth noting that we do not assume that the operator families generated by $A, \{\kappa_1(\vartheta): \vartheta > 0\}$ and $\{\kappa_2(\vartheta): \vartheta > 0\}$, are compact, and this increases the importance of this work.

Recently, Alsaroria and Ghadle (2022) studied the controllability of Problem (1) without delay, assuming that the operator families generated by $A, \{\kappa_1(\vartheta): \vartheta > 0\}$ and , $\{\kappa_2(\vartheta): \vartheta > 0\}$, are compact.

To compare with related work, Alsaroria and Ghadle (2022) examined the controllability of Problem (1) in a special case without delay, assuming the families of operators generated by $A, \{\kappa_1(\vartheta): \vartheta > 0\}$ and $\{\kappa_2(\vartheta): \vartheta > 0\}$ are compact. Johnsona *et al.* (2023) showed the existence of a mild solution to Problem (1) when $Q = 0$. Raja *et al.* (2022) obtained sufficient conditions for approximate control of nonlinear fractional differential integral embeddings of degree $1 < \alpha < 2$, similar to Problem (1) in the case without delay. Wang *et al.* (2015) showed the existence of moderate solutions to Problem (1) when $Q = 0$ and there is no delay. Furthermore, no studies were found on Problem (2).

The main contributions of this paper are summarised as follows:

- A new class of differential inclusions (with the right-hand side as a multi-valued function) generated by sectorial operators with infinite delay, impulsive effects and nonlocal conditions in infinite-dimensional Banach spaces is formulated.
- Unlike other works, such as Alsaroria and Ghadle (2022), this study does not assume that the operator families $\{\kappa_1(\vartheta): \vartheta > 0\}$ and $\{\kappa_2(\vartheta): \vartheta > 0\}$, generated by A, are compact, thereby increasing the importance of the work.
- The discussions are based on the properties of phase spaces, sectorial operators, multi-valued functions, and the Hausdorff measure of noncompactness.
- The existence of mild solutions and the controllability of systems (1) and (2) are established.
- Wang *et al.* (2015) studied Problem (1) in the special case where $Q \equiv 0$ and delay is absent. Alsaroria and Ghadle (2022) addressed Problem (1) with finite delay, assuming the compactness of the operator families $\{\kappa_1(\vartheta): \vartheta > 0\}$ and $\{\kappa_2(\vartheta): \vartheta > 0\}$. This assumption is not adopted in the present study. Therefore, the results generalise Theorem 3.1 in Wang *et al.* (2015) and Theorem 3.4 in Alsaroria and Ghadle (2022).
- The technique presented here can be used to generalise the results obtained in Almarri and Elshenhab (2022), Bedi, Varun *et al.* (2022), Dineshkumar and Udhayakumar (2022), Kumar *et al.* (2022), Salem and Alharbi (2023), and Varun and Bose (2023), particularly when the right-hand side is a multi-valued function instead of a single-valued function and in the presence of instantaneous impulses.
- Finally, an example is provided to demonstrate the applicability of the theoretical results.

For directions on future work, refer to the Discussion and Conclusion section.

Structure of the paper: Section 2 presents the background material required for later development. Section 3 establishes sufficient conditions for the controllability of Problem (1). Section 4 discusses the controllability of Problem (2), based on different sufficient conditions from those in Section 3. Finally, an example is provided.

2. Preliminaries and Notation

Let $P_{ck}(E) = \{Z \subseteq E: Z\ is\ non-empty, convex\ and\ compact\}$, $\mathbb{N} = \{1,2,3, \dots\}$, $\Pi_0 = \{0,1,2, \dots, m\}$ and $\Pi_1 = \{1,2, \dots, m\}$.

Let $D(A) = \{x \in E: A(x)\ is\ defined\}$, $\sigma(A)$ its spectrum, $\rho(A) = \mathbb{C} - \sigma(A)$, and $R(\zeta, A) = (\zeta I - A)^{-1}, \zeta \in \rho(A)$ the resolvent operators of A.

For any function $w: (-\infty, \varpi] \to E$ and $\vartheta \in T$, let $w_\vartheta: (-\infty, 0] \to E$; $w_\vartheta(\tau) = w(\vartheta + \tau); \tau \in (-\infty, 0]$.

Let $T_0 = [0, \vartheta_1], T_i = (\vartheta_i, \vartheta_{i+1}]; i \in \Pi_1$ and consider the Banach space $PC(T, E) = \{w: T \to E: \omega_{|T_i}$ is continuous and $w(\vartheta_i^+)$ and $w(\vartheta_i^-)$ are finite for all $i \in \Pi_0\}$, where $||w||_{PC(T,E)} = sup\{||w(\vartheta)||: \vartheta \in T\}$. Moreover, the Hausdorff measure of noncompactness on $PC(T, E)$ is given by (Cardinali and Rubbioni 2012):

$\chi PC(\Upsilon) = \max_{i=0,1,\dots,m} \chi_{C(\overline{T_i}, E)}(\Upsilon_{|\overline{T_i}})$,

where $\Upsilon \subseteq PC(T, E)$ is bounded,

$\Upsilon_{|\overline{T_i}} = \{f^* \in C(\overline{T_i}, E): f^*(\vartheta) = f(\vartheta);\ \vartheta \in T_i, f^*(\vartheta_i) = f(\vartheta_i^+), f \in \Upsilon\}$,

and $\chi_{C(\overline{T_i}, E)}$ is the Hausdorff measure of noncompactness on $C(\overline{T_i}, E)$ defined by (Kamenskii *et al.*, 2011):

$$\chi_{C(\overline{T_i}, E)}(\Upsilon_{|\overline{T_i}}) = \frac{1}{2} \lim_{\delta \to 0} \sup_{f \in \Upsilon} \max_{|t-s| < \delta} ||f(t) - f(s)||.$$

It is known (example 2.1.3 in (Kamenskii et al., 2011) that if $Z_{|\overline{T_i}}$ is equicontinuous, then

$\chi_{C(\overline{T_i}, E)}(\Upsilon_{|\overline{T_i}}) = sup_{t \in \overline{T_i}} \chi_E\{f(t): f \in \Upsilon_{|\overline{T_i}}\}$,

where χ_E denotes the Hausdorff measure of noncompactness on E.

Definition 1. (Hale and Kato, 1978). A phase space is a vector space β consisting of functions

$w: (-\infty, 0] \to E$, equipped with a seminorm $||.||_\beta$ such that:

1- If $w: (-\infty, \varpi] \to E$ is such that $w_{|T} \in PC(T, E)$ and $w_0 \in \beta$, the following properties hold:

$w_\vartheta \in \beta, \forall \vartheta \in T$.

There exists $C > 0$ such that $||w(\vartheta)|| \leq C||w_\vartheta||_\beta, \forall \vartheta \in T$.

There exist a continuous function $L_1: [0, \infty) \to [0, \infty)$ and a locally bounded function L2: [0,∞)⟶[1,∞) such that

$$(3)\qquad ||\omega_\vartheta||_\beta \leq L_1(\vartheta) sup\{||\omega(\tau)||: \tau \in [0, \vartheta]\} + L_2(\vartheta)||\omega_0||_\beta, \forall \vartheta \in T.$$

2- The function $t \to w_t$ is continuous from T into β.

3- β is complete.

For any $\vartheta \in T = [0, \varpi], the\ fucntion\ w_\vartheta: (-\infty, 0] \to E;\ is\ defined\ by\ w_\vartheta(\tau) = w(\vartheta + \tau)$. .

We now introduce the vector space

$\beta_\varpi := \{w: (-\infty, \varpi] \to E\ such\ that\ w_0 \in \beta, w_{|T} \in PC(T, E)\}$,

endowed with the seminorm $||w||_{\beta_\omega} = ||w_0||_\beta + sup_{\tau \in T}||w(\tau)||$.

Moreover, let

$\mathcal{H} \coloneqq \{w \in \beta_{\varpi}: w_0(\tau) = 0, \forall \tau \in (-\infty, 0]\}$.

It may be noted that $(\mathcal{H}, ||.||_{\mathcal{H}})$ is a Banach space, where $||w||_{\mathcal{H}} = sup_{\vartheta \in T}||w(\vartheta)||$, and the Hausdorff measure of noncompactness on it is defined by:

$\chi_{\mathcal{H}}(D) \coloneqq \max_{i=0,1,2,\ldots,m} \chi_i(D_{|\overline{T_i}})$,

where D is a bounded subset of $\mathcal{H}$.

Remark that, if $\in \mathcal{H}$, then $||w||_{\beta_{\varpi}} = sup_{t \in J}\|w(t)\| = ||w||_{\mathcal{H}}$.

Definition 2. (Kilbas *et al.*, 2006) The Riemann–Liouville fractional integral of order $q > 0$ with the lower limit zero for a function $f \in L^p(T, E), P \in [1, \infty)$ is defined as follows:

$$I^q f(\vartheta) = \frac{1}{\Gamma(q)} \int_0^{\vartheta} (\vartheta - s)^{q-1} f(s)\, ds, \qquad t \in T,$$

where the integration is in the sense of Bochner and Γ is the Euler gamma function.

Definition 3. (Kilbas *et al.*, 2006) Let $q \in (k-1, k)$ and k be a non-negative integer. The Caputo derivative of order q with the lower limit zero for a given function $f \in C^k\ (T, E)$ is defined by

$${}^cD^{\ \gamma} f(\vartheta) = I^{q-k} f(\vartheta) = \frac{1}{\Gamma(q-k)} \int_0^t (\vartheta - s)^{q-k-1} f(s)\, ds.$$

Let us list some properties of the Riemann–liouville integral and Caputo derivative.

Lemma 1. Let $q \in (k-1, k)$ and k be a non-negative integer. The following properties hold.

1. ${}^cD^q(a) = 0$,where a is a constant.
2. If $f \in L^p(T, E), P > \left(\frac{1}{q}\right)$, then ${}^cD^q I^q f(\vartheta) = f(\vartheta), a.e. \vartheta \in T$.
3. ${}^cD^q(\vartheta^{\beta}) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha))} \vartheta^{\beta-\alpha}{}^{\wedge}, \beta > k-1$.
4. If $f \in C^k(T, E)$, then
5. $I^q\ {}^cD^q f(\vartheta) = f(\vartheta) - f(0) - \sum_{n=1}^{n=k-1} \frac{f^{(n)}(0)\vartheta^n}{n!}$.

For more information about the fractional calculus we refer to, (Az-Zo'bi *et al.* 2024), (Al Zubi *et al.* 2024) and (Kilbas *et al.*, 2006).

Next, we recall the notation of fractional sectorial operators (Wang *et al.*, 2015). Let $\emptyset_0 \in (0, \frac{\pi}{2}]$ and $t_0 \in \mathbb{R}$. We denote by $M^{\gamma}(\emptyset_0, t_0)$ to the family of linear closed densely defined operators A satisfying:

$\kappa^{\gamma} \in \rho(A)$ for any $\kappa \in \sum_{\theta_0 + \frac{\pi}{2}}(t_0) = \{\kappa \in \mathbb{C} - \{0\}: |arg(\kappa - t_0)| < \phi_0 + \frac{\pi}{2}\}$.

For any $t > t_0$ and $\emptyset < \emptyset_0$, there is a constant $C = C(t, \emptyset)$ such that $\left\|\kappa^{\gamma-1}R(\kappa^{\gamma}, A)\right\| \le \frac{C}{|\kappa - t|}$, $for\ \lambda \in \kappa \in \sum_{\theta_0 + \frac{\pi}{2}}(t_0)$.

Now, according to Def. 2.20 in (Wang *et al.*, 2015), we have the following definition:

Definition 4. Let $A \in M^{\gamma}(\emptyset_0, t_0), \emptyset_0 \in \left(0, \frac{\pi}{2}\right], t_0 \in \mathbb{R}\ and\ \mho \in L^{\infty}(T, X)$. A function $w^{\mho} \in \beta_{\varpi}$ is called a mild solution for problem (1) if and only if there exists a function $\xi \in L^1(T, E)$ with $\xi(\vartheta) \in \mathcal{F}(\vartheta, (w^{\mho})_{\vartheta}), a.e.$ such that

$$(4) \qquad \omega^u(\vartheta) = \begin{cases} \Psi(\vartheta) - g(w^{\mho})(\vartheta);\ \vartheta \in (-\infty, 0], \\ \kappa_1(\vartheta)\left(\Psi(0) - g(w^{\mho})(0)\right) + \int_0^{\vartheta} \kappa_2(\vartheta - \tau)\left(\xi(\tau) + Q(\mho(\tau))\right) d\tau, v \in T_0, \\ \kappa_1(\vartheta)\left(\Psi(0) - g(w^{\mho})(0)\right) + \kappa_1(\vartheta - \vartheta_1) I_1\left(w^{\mho}(\vartheta_1^-)\right) \\ \quad + \int_0^{\vartheta} \kappa_2(\vartheta - \tau)\left(\xi(\tau) + Q(\mho(\tau))\right) d\tau, v \in T_1, \\ \vdots \\ \kappa_1(\vartheta)\left(\Psi(0) - g(w^{\mho})(0)\right) + \sum_{k=1}^{k=m} \kappa_1(\vartheta - \vartheta_k) I_k\left(w^{\mho}(\vartheta_k^-)\right) \\ \quad + \int_0^{\vartheta} \kappa_2(\vartheta - \tau)\left(\xi(\tau) + Q(\mho(\tau))\right) d\tau, v \in T_m, \end{cases}$$

where

$$\kappa_1(\vartheta) = \frac{1}{2\pi i} \int_{\Upsilon} e^{\xi\vartheta} \zeta^{\gamma-1} R(\zeta^{\gamma-1}, A) d\zeta,$$

$$\kappa_2(\vartheta) = \frac{1}{2\pi i} \int_{\Upsilon} e^{\xi\vartheta} R(\zeta^{\gamma-1}, A) d\zeta,$$

and Υ is a suitable path lying in $\sum_{\theta_0 + \frac{\pi}{2}}(w_0)$.

Lemma 2. (Wang *et al.*, 2015). The operators $\kappa_1(\vartheta)$ and $\kappa_2(\vartheta)$ satisfy the properties:

There are $M \ge 1, w > w_0$ and $C > 0$ such that $||\kappa_1(\vartheta)|| \le Me^{w\vartheta}$ and $||\kappa_2(\vartheta)|| \le Ce^{w\vartheta}(1 + \vartheta^{\gamma-1})$ for each $\vartheta > 0$.

For any $\vartheta \in T$.

$$(5) \qquad ||\kappa_1(\vartheta)||_{L(E)} \le M_1\ and\ ||\kappa_2(\vartheta)||_{L(E)} \le \vartheta^{\gamma-1} M_2,$$

where

$$(6) \qquad M_1 \coloneqq \sup_{0 \le \vartheta \le \varpi} ||\kappa_1(\vartheta)||_{L(E)}\ and\ M_2 \coloneqq \sup_{0 \le \vartheta \le \varpi} Ce^{w\vartheta}(1 + \vartheta^{1-\gamma}).$$

Definition 5. Let $A \in M^{\gamma}(\phi_0, t_0), \phi_0 \in \left(0, \frac{\pi}{2}\right], t_0 \in \mathbb{R}$ and $\in L^p(T, X); p > \frac{1}{\gamma}$. A function $w^z \in \beta_{\omega}$ is called a mild solution for problem (2) if there exists $\xi \in L^1(T, E)$ with $\xi(\vartheta) \in \mathcal{F}(\vartheta, (w^z)_{\vartheta}), a.e.$ such that

$$(7) \qquad \omega^z(\vartheta) = \begin{cases} \Psi(\vartheta) - g(w^z)(\vartheta);\ \vartheta \in (-\infty, 0], \\ \kappa_1(\vartheta)\left(\Psi(0) - g(w^z)(0)\right) + \int_0^{\vartheta} \kappa_2(\vartheta - \tau)(\xi(\tau) + (V_z)(\tau)) d\tau, v \in T_0, \\ \kappa_1(\vartheta)\left(\Psi(0) - g(w^z)(0)\right) + \kappa_1(\vartheta - \vartheta_1) I_1(\vartheta - \vartheta_1^-) \\ \quad + \int_0^{\vartheta} \kappa_2(\vartheta - \tau)(\xi(\tau) + (V_z)(\tau)) d\tau, v \in T_1, \\ \vdots \\ \kappa_1(\vartheta)\left(\Psi(0) - g(w)(0)\right) + \sum_{k=1}^{k=m} \kappa_1(\vartheta - \vartheta_k) I_k\left(w(\vartheta_k^-)\right) \\ \quad + \int_0^{\vartheta} \kappa_2(\vartheta - \tau)(\xi(\tau) + (V_z)(\tau)) d\tau, v \in T_m \end{cases}$$

Definition 6. The Problem (1) is called nonlocal controllable on $T = [0, \varpi]$, if for each $w_1 \in E$, there exists a control function $\mho \in L^{\infty}(T, E)$ such that any corresponding mild solution $w^{\mho} \in \beta_{\varpi}$ to Problem (1) must satisfy $w^{\mho}(0) = \Psi(0) - g(w^{\mho})(0)$ and $w^{\mho}(\varpi) = w_1 - g(w^{\mho})(0)$.

Definition 5. The system (2) is said to be nonlocal controllable on the interval $T = [0, \varpi]$ if for each $w_1 \in E$, there exists a control function $z \in L^p(T, X), p > \frac{1}{\gamma}$ such that any corresponding mild solution $w^z \in \beta_{\varpi}$ for Problem (2) must satisfy $w^z(0) = \Psi(0) - g(w)(0)$ and $w^z(\varpi) = w_1 - g(w^z)(0)$.

Lemma 3. (O'Regan and Precup, 2000, Theorem 3.1). Let D be a closed convex subset of a Banach space and $R: D \to \wp_c(D)$ with a closed graph and maps compact sets into relatively compact sets. Assume that there is $w_0 \in D$ such that, for any $\subseteq D$, with $K = conv(\{w_0\} \cup R(K)), \overline{K} = \bar{Z}$, $Z \subseteq D$ countable, we get K is relatively compact. Then R has a fixed point.

2.1. Controllability of the system (1):

Theorem 1. Assume the following conditions:

$(H\mathcal{F})$ $\mathcal{F}: T \times \beta \to P_{ck}(E)$ such that

$(H\mathcal{F}_1)$ For each $z \in \beta$, the multifunction $\vartheta \to \mathcal{F}(\vartheta, z)$ admits a strongly measurable selection, and for almost each $\vartheta \in T$, the multifunction $z \to \mathcal{F}(\vartheta, z)$ is upper semicontinuous.

$(H\mathcal{F}_2)$ For any $n \in \mathbb{N}$, there exists a $\varrho_n \in L^p(T, \mathbb{R}^+)$ satisfying $sup_{||w||_{\beta \le n}}||\mathcal{F}(\vartheta, w)|| \le \varrho_n(\vartheta)$, for $a.e. \vartheta \in T$ and

(8) $$\liminf_{n\to\infty}\frac{||\varrho_n||_{L^p(T,\mathbb{R}^+)}}{n}=0.$$

$(H\mathcal{F}_3)$ There is a $\varsigma\in L^p(T,E), p>\frac{1}{\gamma}$ such that, for each bounded subset $Z\subset\beta$ we have

(9) $$\chi_E(\mathcal{F}(\tau,Z))\le\varsigma(\tau)\sup_{\vartheta\in(-\infty,0]}\chi_E\{\psi(\vartheta):\psi\in Z\},$$

$a.e. for\ \tau\in T$.

(H_g) The function $g:\beta_\varpi\to\beta$ verifies:

(Hg_1)

(10) $$\liminf_{||w||_{\beta_\varpi}\to\infty}\frac{||g(w)(0)||_E}{||w||_{\beta_\varpi}}=0.$$

(Hg_2) If $w_n\to w\ in\ \beta_\varpi$, then $\lim_{n\to\infty}g(w)(0)_E=g(w)(0)$.

(Hg_3) $\overline{\{g(z)(0):z\in D\}}$ is compact in E whenever $D\subseteq\beta_\varpi$ is bounded.

(HI) For any $i\in\Pi_1, I_i:E\to E$ is continuous and compact, and there are non-decreasing functions $h_i:\mathbb{R}^+\to\mathbb{R}$ such that $||I_i(w)||\le h_i(||w||), w\in E$ and

(11) $$\liminf_{n\to\infty}\frac{|h_i(n)|}{n}=0, i\in\Pi_1.$$

$(H\Delta)$ The operator $\Delta:L^\infty(T,X)\to E$, defined by

(12) $$\Delta(\mho)=\int_0^\varpi\kappa_2(\varpi-\tau)Q(\mho(\tau))d\tau,$$

is linear, bounded and has a bounded inverse Δ^{-1}.

Then, the system (1.1) is controllable on T if the following inequality is satisfied:

(13) $$2M_{2^\eta}||\beta||_{L^p(T,\mathbb{R}^+)}\left[1+N^2M_2\frac{\varpi^\gamma}{\gamma}\right]<1,$$

where $\eta=\left(\frac{p-1}{p\gamma-1}\right)^{\frac{p-1}{p}}\varpi^{\gamma-\frac{1}{p}}$ and $N>0$ with $||\Delta^{-1}||\le N$ and $||Q||\le N$.

Proof. Note that, for any $\mho\in L^\infty(T,X)$,

$$||\Delta(\mho)||\le M_2N\int_0^\varpi(\varpi-\tau)^{\gamma-1}||\mho(\tau)||d\tau$$
$$\le M_2N||\mho||_{L^\infty(T,\omega)}\frac{\varpi^\gamma}{\gamma},$$

which means that Δ is well defined. Let $w\in\beta$ be fixed. In view of $(H\mathcal{F}_1)$ and $(H\mathcal{F}_2)$, there is $\xi\in L^p(T,E)$ with $\xi(\vartheta)\in\mathcal{F}(\vartheta,w_\vartheta), a.e.$ Then, thanks to $(H\Delta)$ we can define a control function $\mho_{w,\xi}\in L^\infty(T,\omega)$ as:

(14) $$\mho_{w,\xi}=\Delta^{-1}[w_1g(w)(0)-\kappa_1(\varpi)(\Psi(0)-g(w)(0))-\sum_{k=1}^{k=m}\kappa_1(\varpi-\vartheta_k)I_k(w(\vartheta_k^-))-\int_0^\varpi\kappa_2(\varpi-\tau)\xi(\tau)d\tau].$$

Thus, a multifunction $R:\mathcal{H}\to 2^{\mathcal{H}}$ can be defined as follows: $y\in R(w)$ if and only if

(15) $$y(\vartheta)=\begin{cases}0,\vartheta\in(-\infty,0],\\ \kappa_1(\vartheta)(\Psi(0)-g(w)(0))+\int_0^\vartheta\kappa_2(\vartheta-\tau)\left(\xi(\tau)+Q\left(\mho_{w,\xi}(\tau)\right)\right)d\tau, v\in T_0,\\ \kappa_1(\vartheta)(\Psi(0)-g(\omega)(0))+\kappa_1(\vartheta-\vartheta_1)I_1\left(w^{\mho}(\vartheta_1^-)\right)\\ \quad+\int_0^\vartheta\kappa_2(\vartheta-\tau)\left(\xi(\tau)+Q\left(\mho_{w,\xi}(\tau)\right)\right)d\tau, v\in T_1,\\ \vdots\\ \kappa_1(\vartheta)(\Psi(0)-g(w)(0))+\sum_{k=1}^{k=m}\kappa_1(\vartheta-\vartheta_k)I_k\left(w^{\mho}(\vartheta_k^-)\right)\\ \quad+\int_0^\vartheta\kappa_2(\vartheta-\tau)\left(\xi(\tau)+Q\left(\mho_{w,\xi}(\tau)\right)\right)d\tau, v\in T_m,\end{cases}$$

where $\xi\in\tau^1_{\mathcal{F}(.,w(.))}=\{v\in L^p(T,E):v(t)\in\mathcal{F}(t,w_t),a.e.\}$. Notice that, if W is a fixed point for R, then from $(H\Delta)$, (14) and (15)

$$\begin{aligned}\omega(\varpi)&=\kappa_1(\varpi)(\Psi(0)-g(w)(0))\\&\quad+\sum_{k=1}^{k=m}\kappa_1(\varpi-\vartheta_k)I_k(w(\vartheta_k^-))\\&\quad+\int_0^\varpi\kappa_2(\varpi-\tau)\xi(\tau)d\tau\\&\quad+\int_0^\varpi\kappa_2(\varpi-\tau)Q\left(\mho_{w,\xi}(\tau)\right)d\tau\\&=\kappa_1(\varpi)(\Psi(0)-g(w)(0))+\sum_{k=1}^{k=m}\kappa_1(\varpi-\vartheta_k)I_k(w(\vartheta_k^-))\\&\quad+\int_0^\varpi\kappa_2(\varpi-\tau)\xi(\tau)d\tau+\Delta(\mho_{w,\xi})\\&=\kappa_1(\varpi)(\Psi(0)-g(w)(0))+\sum_{k=1}^{k=m}\kappa_1(\varpi-\vartheta_k)I_k(w(\vartheta_k^-))\\&\quad+\int_0^\varpi\kappa_2(\varpi-\tau)\xi(\tau)d\tau+w_1-g(w)(0)\\&\quad-\kappa_1(\varpi)(\Psi(0)-g(w)(0))\\&\quad-\sum_{k=1}^{k=m}\kappa_1(\varpi-\vartheta_k)I_k(w(\vartheta_k^-))\\&\quad-\int_0^\varpi\kappa_2(\varpi-\tau)\xi(\tau)d\tau=w_1-g(w)(0).\end{aligned}$$

Thus, the function $\overline{w}:(-\infty,\varpi]\to E$ defined by

$$\overline{w}(\vartheta)=\begin{cases}\Psi(\vartheta)-g(w)(\vartheta),\vartheta\in(-\infty,0],\\ w(\vartheta),\vartheta\in[0,\varpi],\end{cases}$$

is a mild solution of system (1) and satisfies $\overline{w}(0)=\Psi(0)-g(w)(0)$ and $\overline{w}(\varpi)=w_1-g(w)(0)$.

Our goal is to prove, using Lemma 2, that R has a fixed point. Clearly $R(w); w\in\mathcal{H}$ is convex.

Step1. In this step, we demonstrate the existence of $n_0\in\mathbb{N}$ such that $R(D_{n_0})\subseteq D_{n_0}; D_{n_0}=\{w\in\mathcal{H}:||w||_{\mathcal{H}}\le n_0$. Assume, by contradiction, that for each $r\in\mathbb{N}$, there exist $w_r,k_r\in\mathcal{H}$ with $k_r\in R(w_r)$, $||w_r||_{\mathcal{H}}\le r$ and $||k_r||_{\mathcal{H}}>0$. So, there exists a sequence $(\xi_r)_{r\ge1}\in\tau^p_{\mathcal{F}(.,(w_r)_\vartheta)}$ such that:

(16) $$k_r(\vartheta)=\begin{cases}0,\vartheta\in(-\infty,0],\\ \kappa_1(\vartheta)(\Psi(0)-g(w_r)(0))+\int_0^\vartheta\kappa_2(\vartheta-\tau)\left(\xi_r(\tau)+Q\left(\mho_{w_r,\xi_r}(\tau)\right)\right)d\tau, v\in T_0,\\ \kappa_1(\vartheta)(\Psi(0)-g(w_r)(0))+\kappa_1(\vartheta-\vartheta_1)I_1(w_r(\vartheta_1^-))\\ \quad+\int_0^\vartheta\kappa_2(\vartheta-\tau)\left(\xi_r(\tau)+Q\left(\mho_{w_r,\xi_r}(\tau)\right)\right)d\tau, v\in T_1,\\ \vdots\\ \kappa_1(\vartheta)(\Psi(0)-g(w_r)(0))+\sum_{k=1}^{k=m}\kappa_1(\vartheta-\vartheta_k)I_k(w_r(\vartheta_k^-))\\ \quad+\int_0^\vartheta\kappa_2(\vartheta-\tau)\left(\xi_r(\tau)+Q\left(\mho_{w_r,\xi_r}(\tau)\right)\right)d\tau, v\in T_m,\end{cases}$$

Notably, (3) yields $||(w_r)_\vartheta||_\beta\le\xi r,\forall\vartheta\in T\ and\ r\ge1$. Then, by $(H\mathcal{F}_2)$ for all $r\ge1$,

(17) $$||\xi_r(\tau)||\le\xi\varphi_r(\tau), a.e.\vartheta\in T,$$

where $\xi=\sup\{L_1(\vartheta):\vartheta\in T\}$. Since $p>\frac{1}{\gamma}$, the function $\tau\to(\vartheta-\tau)^{\gamma-1}$ belongs to $L^{\frac{p}{p-1}}(T,\mathbb{R}^+)$. Consequently, using (17) and Holder's inequality, we obtain, for any $r\in\mathbb{N}$ and $\vartheta\in T$

$$
\begin{aligned}
||\int_0^{\vartheta}\kappa_2(\vartheta-\tau)\xi_r(\tau)d\tau|| &\le \xi M_2\int_0^{\vartheta}(\vartheta-\tau)^{\gamma-1}\varphi_r(\tau)d\tau\\
&\le \xi M_2||\varphi_r||_{L^p(T,\mathbb{R}^+)}\left(\int_0^{\vartheta}(\vartheta-\tau)^{\frac{(\gamma-1)p}{p-1}}d\tau\right)^{\frac{p-1}{p}}\\
&= \xi M_2||\varphi_r||_{L^p(T,\mathbb{R}^+)}\left(\frac{p-1}{p\gamma-1}\right)^{\frac{p-1}{p}}\vartheta^{\gamma-\frac{1}{p}}\\
&\le \xi M_2\eta||\varphi_r||_{L^p(T,\mathbb{R}^+)}.
\end{aligned}
\tag{18}
$$

Next, since $\mho_{w,\xi}\in L^{\infty}(T,w)$, then for any $w\in\mathcal{H}$ and any $\xi\in\tau^{p}_{\mathcal{F}(.,w(.))}$, we have, for almost every $\tau\in T$

$$
\begin{aligned}
||\mho_{w,\xi}(\tau)|| &\le ||\mho_{w,\xi}||_{L^{\infty}(T,X)}\\
&\le ||\Delta^{-1}||[||w_1||+||g(w)(0)||+||\kappa_1(\varpi)||(||\Psi(0)||+||g(w)(0)||)\\
&+\sum_{k=1}^{k=m}||\kappa_1(\varpi-\vartheta_k)||||I_k(w(\vartheta_k^-))||\\
&+||\int_0^{\varpi}\kappa_2(\varpi-\tau)\xi(\tau)d\tau||].
\end{aligned}
\tag{19}
$$

This inequality gives us,

$$
\begin{aligned}
||\int_0^{\vartheta}\kappa_2(\vartheta-\tau)Q(\mho_{w_r,\xi_r}(\tau))d\tau|| &\le M_2N\int_0^{\vartheta}(\vartheta-\tau)^{\gamma-1}||\mho_{w_r,\xi_r}(\tau)||d\tau\\
&\le M_2N\frac{\varpi^{\gamma}}{\gamma}N[||w_1||+||g(w_r)(0)||+||\kappa_1(\varpi)||(||\Psi(0)||+||g(w_r)(0)||)\\
&+\sum_{k=1}^{k=m}||\kappa_1(\varpi-\vartheta_k)||||I_k(w_r(\vartheta_k^-))||\\
&+||\int_0^{\varpi}\kappa_2(\varpi-\tau)\xi_r(\tau)d\tau||].
\end{aligned}
\tag{20}
$$

Consequently, if $\vartheta\in T_0$, then from (Hg_1), (HI), (16) and (20), it follows:

$$
\begin{aligned}
||\kappa_r(\vartheta)|| &\le M_1(||\Psi(0)||+||g(w_r)(0)||)+M_2\xi||\varphi_r||_{L^p(T,\mathbb{R}^+)}\eta\\
&+M_2N\frac{\varpi^{\gamma}}{\gamma}N[||w_1||+||g(w_r)(0)||+M_1(||\Psi(0)||+||g(w_r)(0)||)\\
&+M_1\sum_{k=1}^{k=m}h_k(||w_r||)+M_2\xi||\varphi_r||_{L^p(T,\mathbb{R}^+)}\eta].
\end{aligned}
$$

Likewise, when $\vartheta\in T_{\xi},\xi\in\Pi_1$, we have:

$$
\begin{aligned}
||\kappa_r(\vartheta)|| &\le M_1(||\Psi(0)||+||g(w_r)(0)||)+M_1\sum_{k=1}^{k=m}h_k(||w_r||)+\eta M_2\xi||\varphi_r||_{L^p(T,\mathbb{R}^+)}\\
&+\frac{\varpi^{\gamma}}{\gamma}N^2M_2[||w_1||+||g(w_r)(0)||+M_1(||\Psi(0)||+||g(w_r)(0)||)\\
&+M_1\sum_{k=1}^{k=m}h_k(||w_r||)+\eta M_2\xi||\varphi_r||_{L^p(T,\mathbb{R}^+)}].
\end{aligned}
$$

Then,

$$
\begin{aligned}
r<||\kappa_r|| &\le M_1(||\Psi(0)||+||g(w_r)(0)||)+M_1\sum_{k=1}^{k=m}h_k(r)+\eta M_2\xi||\varphi_r||_{L^p(T,\mathbb{R}^+)}\\
&+\frac{\varpi^{\gamma}}{\gamma}N^2M_2[||w_1||+||g(w_r)(0)||+M_1(||\Psi(0)||+||g(w_r)(0)||)\\
&+M_1\sum_{k=1}^{k=m}h_k(r)+\eta M_2\xi||\varphi_r||_{L^p(T,\mathbb{R}^+)}].
\end{aligned}
$$

By dividing both sides by r and taking the $\lim inf$ as $r\to\infty$, and using (8), (10), and (11), we obtain the contradiction 1<0. Thus, there exists $n_0\in\mathbb{N}$ such that $R(D_{n_0})\subseteq D_{n_0}$.

Step 2. Let $K=R(D_{n_0})$. We claim, in this step, that the set $K_{|\overline{T_i}}$ is equicontinuous for each $i\in\Pi_m$, where

$K_{|\overline{T_i}}=\{y^*\in C(\overline{T}_i,E): y^*(\vartheta)=y(\vartheta),\vartheta\in T_i, y^*(\vartheta_i)=y(\vartheta_i^+), y\in K\}$.

Let $y\in K$, Then, there exists $w\in D_{n_0}$ such that $y\in R(w)$. Hence, there exists $\xi\in\tau^{p}_{\mathcal{F}(.,w_{\vartheta})}$ such that

$$
y(\vartheta)=\begin{cases}
0,\vartheta\in(-\infty,0],\\
\kappa_1(\vartheta)(\Psi(0)-g(w)(0))+\int_0^{\vartheta}\kappa_2(\vartheta-\tau)\left(\xi(\tau)+Q\left(\mho_{w,\xi}(\tau)\right)\right)d\tau, v\in T_0,\\
\kappa_1(\vartheta)(\Psi(0)-g(w)(0))+\kappa_1(\vartheta-\vartheta_1)I_1(w(\vartheta_1^-))\\
\quad+\int_0^{\vartheta}\kappa_2(\vartheta-\tau)\left(\xi(\tau)+Q\left(\mho_{w,\xi}(\tau)\right)\right)d\tau, v\in T_1,\\
\quad\vdots\\
\kappa_1(\vartheta)(\Psi(0)-g(w)(0))+\sum_{k=1}^{k=m}\kappa_1(\vartheta-\vartheta_k)I_k(w(\vartheta_k^-))\\
\quad+\int_0^{\vartheta}\kappa_2(\vartheta-\tau)\left(\xi(\tau)+Q\left(\mho_{w,\xi}(\tau)\right)\right)d\tau, v\in T_m,
\end{cases}
$$

Case 1. Let $\vartheta,\vartheta+\delta\in\overline{T_0}=T_0$. Then,

$$
\begin{aligned}
||y^*(\vartheta+\delta)-y^*(\vartheta)|| &= ||y(\vartheta+\delta)-y(\vartheta)||\\
&\le ||\kappa_1(\vartheta+\delta)(\Psi(0)-g(w)(0))-\kappa_1(\vartheta)(w_0-g(w)(0))||\\
&+||\int_0^{\vartheta+\delta}\kappa_2(\vartheta+\delta-\tau)\left(\xi(\tau)+Q\left(\mho_{w,\xi}(\tau)\right)\right)d\tau\\
&-\int_0^{\vartheta}\kappa_2(\vartheta-\tau)\left(\xi(\tau)+Q\left(\mho_{w,\xi}(\tau)\right)\right)d\tau\\
&+||\kappa_1(\vartheta+\delta)(\Psi(0)-g(w)(0))-\kappa_1(\vartheta)(w_0-g(w)(0))||\\
&+||\int_{\vartheta}^{\vartheta+\delta}\kappa_2(\vartheta+\delta-\tau)\left(\xi(\tau)+Q\left(\mho_{w,\xi}(\tau)\right)\right)d\tau||\\
&+||\int_0^{\vartheta}[\kappa_2(\vartheta+\delta-\tau)-\kappa_2(\vartheta-\tau)]\left(\xi(\tau)+Q\left(\mho_{w,\xi}(\tau)\right)\right)d\tau||\\
&= G_1+G_2+G_3,
\end{aligned}
$$

where

$G_1=||\kappa_1(\vartheta+\delta)(w_0-g(w)(0))-\kappa_1(\vartheta)(w_0-g(w)(0))||$,

$G_2=||\int_{\vartheta}^{\vartheta+\delta}\kappa_2(\vartheta+\delta-\tau)\left(\xi(\tau)+Q\left(\mho_{w,\xi}(\tau)\right)\right)d\tau||$,

and

$G_3=||\int_0^{\vartheta}[\kappa_2(\vartheta+\delta-\tau)-\kappa_2(\vartheta-\tau)]\left(\xi(\tau)+Q\left(\mho_{w,\xi}(\tau)\right)\right)d\tau||$.

Thanks to (Hg_3), there exists a constant $k>0$ such that $||g(w)(0)||\le k,\forall w\in B_{n_0}$. Thus,

$$
\begin{aligned}
\lim_{\delta\to0}G_1 &= \lim_{\delta\to0}||\kappa_1(\vartheta+\delta)(\Psi(0)-g(w)(0))-\kappa_1(\vartheta)(\Psi(0)-g(w)(0))||\\
&\le ||\Psi(0)-g(w)(0)||\lim_{\delta\to0}||\kappa_1(\vartheta+\delta)-\kappa_1(\vartheta)||\\
&\le (||\Psi(0)||+k)\lim_{\delta\to0}||\kappa_1(\vartheta+\delta)-\kappa_1(\vartheta)||=0,
\end{aligned}
$$

and this limit is independent of the choice of $w\in B_{n_0}$.

Next, by (19) and Hölder's inequality, it follows that:

$$
\lim_{\delta\to0}G_2\le M_2\lim_{\delta\to0}\int_{\vartheta}^{\vartheta+\delta}(\vartheta+\delta-\tau)^{\gamma-1}(||\xi(\tau)||+||Q||||\mho_{w,\xi}(\tau)||)d\tau
$$

$$\leq M_2[\lim_{\delta\to 0}\int_{\vartheta}^{\vartheta+\delta}(\vartheta+\delta-\tau)^{\gamma-1}\varphi_{n_0}(\tau)d\tau + N\lim_{\delta\to 0}\int_{\vartheta}^{\vartheta+\delta}(\vartheta+\delta-\tau)^{\gamma-1}||\mho_{w,\xi}(\tau)||d\tau]$$

$$\leq M_2||\varphi_{n_0}||_{L^p(T,\mathbb{R}^+)}\lim_{\delta\to 0}\left(\int_{\vartheta}^{\vartheta+\delta}(\vartheta+\delta-\tau)^{\frac{p(\gamma-1)}{p-1}}d\tau\right)^{\frac{p-1}{p}}$$

$$+M_2N||\mho_{w,\xi}||_{L^\infty(T,w)}\lim_{\delta\to 0}\int_{\vartheta}^{\vartheta+\delta}(\vartheta+\delta-\tau)^{\gamma-1}d\tau=0,$$

and this limit is independent of the choice of $w\in B_{n_0}$.

For G_3, note that for almost every $\vartheta\in T, ||\xi(\vartheta)||\leq\varphi_{n_0}(\vartheta)$. Moreover, from (19), for almost every $\vartheta\in T$, we obtain:

$$\left\|Q\left(\mho_{w,\xi}(\vartheta)\right)\right\|\leq||Q||\left\|\mho_{w,\xi}(\vartheta)\right\|$$

$$\leq N^2[||w_1||+||g(w)(0)||+M_1(||w_0||+||g(w)(0)||)$$

(21) $$+M_1\sum_{k=1}^{k=m}h_k(||w||)+\int_0^{\vartheta}\kappa_2(\vartheta-\tau)\xi(\tau)d\tau]$$

$$\leq N^2[||w_1||+k+M_1(||w_0||+k)+M_1\sum_{k=1}^{k=m}h_k(n_0)+M_2\eta\left\|\varphi_{n_0}\right\|_{L^p(T,\mathbb{R}^+)}]$$

Then, in view of (21), we have:

$$\lim_{\delta\to 0}G_3\leq\int_0^{\vartheta}\lim_{\delta\to 0}||\,\kappa_2(\vartheta+\delta-\tau)-\kappa_2(\vartheta-\tau)]\xi(\tau)d\tau||$$

$$+N\int_0^{\vartheta}\lim_{\delta\to 0}||[\,\kappa_2(\vartheta+\delta-\tau)-\kappa_2(\vartheta-\tau)]Q\left(\mho_{w,\xi}(\vartheta)\right)||d\tau$$

In view of the definition of κ_2, we conclude that $\lim_{\delta\to 0}G_3=0$, independently of the choice of W.

Case 2. Let $\vartheta,\vartheta+\delta\in T_i, i\in\Pi_1$.

$$||y^*(\vartheta+\delta)-y^*(\vartheta)||=||y(\vartheta+\delta)-y(\vartheta)||$$

$$\leq||\kappa_1(\vartheta+\delta)(\Psi(0)-g(w)(0))-\kappa_1(\vartheta)(\Psi(0)-g(w)(0))||$$

$$+\sum_{k=1}^{k=i}||\kappa_1(\vartheta+\delta-\vartheta_k)-\kappa_1(\vartheta-\vartheta_k)||h_k(n_0)||$$

$$+||\int_0^{\vartheta+\delta}\kappa_2(\vartheta+\delta-\tau)\left(\xi(\tau)+Q\left(\mho_{w,\xi}(\tau)\right)\right)d\tau-\int_0^{\vartheta}\kappa_2(\vartheta-\tau)\left(\xi(\tau)+Q\left(\mho_{w,\xi}(\tau)\right)\right)d\tau$$

As in the previous case, we get $\lim_{\delta\to 0}||y(\vartheta+\delta)-y(\vartheta)||=0$.

Case 3. Let $\vartheta=\vartheta_i$; $i\in\Pi_1,\delta>0$ and $\tau>0$ such that $\vartheta_i+\delta\in T_i$ and $\vartheta_i<\tau<\vartheta_i+\delta\leq\vartheta_{i+1}$. Then,

$$||y^*(\vartheta_i+\delta)-y^*(\vartheta_i)||=\lim_{\tau\to\vartheta_i^+}||y(\vartheta_i+\delta)-y(\tau)||.$$

Note that,

$$||y(\vartheta_i+\delta)-y(\tau)||$$

$$\leq||\kappa_1(\vartheta_i+\delta)(\Psi(0)-g(w)(0))-\kappa_1(\tau)(\Psi(0)-g(w)(0))||$$

$$+\sum_{k=1}^{k=T}||\kappa_1(\vartheta_i+\delta-\vartheta_k)I_k(w(\vartheta_k^-))-\kappa_1(\lambda-\vartheta_k)I_k(w(\vartheta_k^-))||$$

$$+||\int_0^{\vartheta_i+\delta}\kappa_2(\vartheta_i+\delta-\tau)\left(\xi(\tau)+Q\left(\mho_{w,\xi}(\tau)\right)\right)d\tau-\int_0^{\vartheta_i}\kappa_2(\tau-\tau)\left(\xi(\tau)+Q\left(\mho_{w,\xi}(\tau)\right)\right)d\tau.$$

As in the first case, we get $\lim_{\substack{\delta\to 0\\ \tau\to\vartheta_i^+}}||y(\vartheta_i^+ +\delta)-y(\tau)||=0$. As a consequence of this discussion, the set $K_{|\overline{T_i}}$ is Equi continuous for every $i\in\Pi_0$.

Step 3. The graph of $R_{|B_{n_0}}$ is closed.

Let $w_n\to w$ in B_{n_0} and $z_n\in R(w_n)$ with $z_n\to z\ in\ \mathcal{H}$. We must show that $z\in R(w)$. For each $n\geq 1$, let $\xi_n\in\tau^p_{\xi(.,(w_n)_\vartheta)}$ be such that:

$$z_n(\vartheta)=\begin{cases}0,\vartheta\in(-\infty,0],\\ \kappa_1(\vartheta)(\Psi(0)-g(w_n)(0))+\int_0^{\vartheta}\kappa_2(\vartheta-\tau)\left(\xi_n(\tau)+Q\left(\mho_{w_n,\xi_n}(\tau)\right)\right)d\tau, v\in T_0,\\ \kappa_1(\vartheta)(\Psi(0)-g(w_n)(0))+\kappa_1(\vartheta-\vartheta_1)I_1(w_n(\vartheta_1^-))\\ \quad+\int_0^{\vartheta}\kappa_2(\vartheta-\tau)\left(\xi_n(\tau)+Q\left(\mho_{w_n,\xi_n}(\tau)\right)\right)d\tau, v\in T_1,\\ \vdots\\ \kappa_1(\vartheta)(\Psi(0)-g(w_n)(0))+\sum_{k=1}^{k=m}\kappa_1(\vartheta-\vartheta_k)I_k(w_n(\vartheta_k^-))\\ \quad+\int_0^{\vartheta}\kappa_2(\vartheta-\tau)\left(\xi_n(\tau)+Q\left(\mho_{w_n,\xi_n}(\tau)\right)\right)d\tau, v\in T_m,\end{cases}$$

In view of $(H\mathcal{F}_2)$, $||\xi_n(\vartheta)||\leq\xi\varphi_{n_0}(\vartheta),\forall n\geq 1$, and for $a.e.\vartheta\in T$. Hence, the set $\{\xi_n: n\geq 1\}$ is weakly compact in $L^p(T,E)$. Since $p>1$, we can assume that ξ_n converges weakly to a function $\xi\in L^p(T,E)$. Thanks to Mazur's lemma, there exists a sequence $(\varsigma_n)_{n\geq 1}$, where each ς_n is a convex combination of $\xi_j\ with\ j\geq n$, and $\varsigma_n\to\xi$ strongly in $L^1(T,E)$. Let

$$\overline{z_n}(\vartheta)=\begin{cases}0,\vartheta\in(-\infty,0],\\ \kappa_1(\vartheta)(\Psi(0)-g(w_n)(0))+\int_0^{\vartheta}\kappa_2(\vartheta-\tau)\left(\varsigma_n(\tau)+Q\left(\mho_{w_n,z_n}(\tau)\right)\right)d\tau, v\in T_0,\\ \kappa_1(\vartheta)(\Psi(0)-g(w_n)(0))+\kappa_1(\vartheta-\vartheta_1)I_1(w_n(\vartheta_1^-))\\ \quad+\int_0^{\vartheta}\kappa_2(\vartheta-\tau)\left(\varsigma_n(\tau)+Q\left(\mho_{w_n,z_n}(\tau)\right)\right)d\tau, v\in T_1,\\ \vdots\\ \kappa_1(\vartheta)(w_0-g(w_n)(0))+\sum_{k=1}^{k=m}\kappa_1(\vartheta-\vartheta_k)I_k(w_n(\vartheta_k^-))\\ \quad+\int_0^{\vartheta}\kappa_2(\vartheta-\tau)\left(\varsigma_n(\tau)+Q\left(\mho_{w_n,z_n}(\tau)\right)\right)d\tau, v\in T_m,\end{cases}$$

Note that, $\overline{z_n}\to z$ and for each $\vartheta\in T$, and each $\tau\in(0,\vartheta]$, we get:

$||\kappa_2(\vartheta-\tau)\varsigma_n(\tau)||\leq M_2(\vartheta-\tau)^{\gamma-1}\varphi_{n_0}(\tau);\ \forall n\geq 1$.

Because κ_2 is continuous, it follows that,

(22) $$\lim_{n\to\infty}\int_0^{\vartheta}\kappa_2(\vartheta-\tau)\varsigma_n(\tau)d\tau=\int_0^{\vartheta}\kappa_2(\vartheta-\tau)\xi(\tau)d\tau.$$

From (22), (Hg_2), the continuity of Δ^{-1},κ_1 and (14), we obtain $\lim_{n\to\infty}\mho_{w_n,z_n}=\mho_{w,\xi}$, in $L^\infty(T,w)$. So, $\lim_{n\to\infty}\mho_{w_n,z_n}(\vartheta)=\mho_{w,\xi}(\vartheta), a.e.\vartheta\in T$. Because Q is continuous, it follows that $\lim_{n\to\infty}Q(\mho_{w_n,z_n}(\vartheta))=Q\left(\mho_{w,\xi}(\vartheta)\right), a.e.\vartheta\in T, in\ E$. Therefore, by the Lebesgue dominated convergence theorem, we conclude,

$$z(\vartheta) = \begin{cases} 0, \vartheta \in (-\infty, 0], \\ \kappa_1(\vartheta)(\Psi(0) - g(w)(0)) + \int_0^\vartheta \kappa_2(\vartheta - \tau)(\xi(\tau) + Q(\mho_{w,\xi}(\tau)))d\tau, v \in T_0, \\ \kappa_1(\vartheta)(\Psi(0) - g(w)(0)) + \kappa_1(\vartheta - \vartheta_1)I_1(w(\vartheta_1^-)) \\ \quad + \int_0^\vartheta \kappa_2(\vartheta - \tau)\left(\xi(\tau) + Q\left(\mho_{w,\xi}(\tau)\right)\right)d\tau, v \in T_1, \\ \vdots \\ \kappa_1(\vartheta)(\Psi(0) - g(w_n)(0)) + \sum_{k=1}^{k=m} \kappa_1(\vartheta - \vartheta_k)I_k(w(\vartheta_k^-)) \\ \quad + \int_0^\vartheta \kappa_2(\vartheta - \tau)\left(\xi(\tau) + Q\left(\mho_{w,\xi}(\tau)\right)\right)d\tau, v \in T_m, \end{cases}$$

Notice that, thanks to (3) it follows for $\vartheta \in T$ that

$$\lim_{n\to\infty} ||(w_n)_\vartheta - w_\vartheta||_B = \lim_{n\to\infty} ||(w_n - w)_\vartheta||_B \le \xi \lim_{n\to\infty} ||w_n - w||_{\mathcal{H}} = 0.$$

This implies with the upper semi continuity of $\mathcal{F}(\vartheta,.); a.e. \vartheta \in T, \xi(\vartheta) \in \mathcal{F}(\vartheta, w_\vartheta), a.e. \vartheta \in T$. This proves that the graph of R is closed.

Step 4. Let $K \subseteq B_{n_0}$ *be defined as* $K = conv(\{0\} \cup R(K))$, *with* $\bar{K} = \bar{B}$ where $B \subseteq K$ is countable. We claim that K is relatively compact. From Step 2, we know that K is equicontinuous on each $\overline{T}_i; i \in \Pi_0$. It remains to show that the set $\{K(\vartheta);\ \vartheta \in T\}$ is relatively compact in E. Since $B \subseteq K = conv(\{0\} \cup R(K))$ is countable, we can find a countable set $H = \{y_n : n \ge 1\} \subseteq R(K)$ such that $B \subseteq conv(\{0\} \cup H)$. Then, for any $\vartheta \in T$,

$$\chi(K(\vartheta)) = \chi(\bar{B}(\vartheta))\chi(H(\vartheta)) = \chi\{y_n(\vartheta) : n \ge 1\}. \quad (23)$$

Now, for any $n \ge 1$, let $w_n \in K \subseteq B_{n_0}$ with $y_n \in R(w_n)$. Then, there exists $\xi_n(\vartheta) \in \tau^p_{\mathcal{F}(\vartheta,(w_n)_\vartheta)}$ such that $||\xi_n(\vartheta)|| \le \varphi_{n_0}(\vartheta), a.e., \forall n \ge 1$. Moreover,

$$y_n(\vartheta) = \begin{cases} 0, \vartheta \in (-\infty, 0], \\ \kappa_1(\vartheta)(w_0 - g(w)(0)) + \int_0^\vartheta \kappa_2(\vartheta - \tau)(\xi_n(\tau) + Q\left(\mho_{w_n,\xi_n}(\tau)\right))d\tau, v \in T_0, \\ \kappa_1(\vartheta)(w_0 - g(w)(0)) + \kappa_1(\vartheta - \vartheta_1)I_1(w(\vartheta_1^-)) \\ \quad + \int_0^\vartheta \kappa_2(\vartheta - \tau)\left(\xi_n(\tau) + Q\left(\mho_{w_n,\xi_n}(\tau)\right)\right)d\tau, v \in T_1, \\ \vdots \\ \kappa_1(\vartheta)(w_0 - g(w)(0)) + \sum_{k=1}^{k=m} \kappa_1(\vartheta - \vartheta_k)I_k(w(\vartheta_k^-)) \\ \quad + \int_0^\vartheta \kappa_2(\vartheta - \tau)\left(\xi_n(\tau) + Q\left(\mho_{w_n,\xi_n}(\tau)\right)\right)d\tau, v \in T_m. \end{cases}$$

Then, by using the properties of the measure of noncompactness (Kamenskii *et al.*, 2011) we obtain:

$$\chi\{y_n(\vartheta) : n \ge 1\} \le \begin{cases} 0, \vartheta \in (-\infty, 0], \\ \chi\{\kappa_1(\vartheta)(w_0 - g(w_n)(0)) : n \ge 1\} \\ + \chi\left\{\int_0^\vartheta \kappa_2(\vartheta - \tau)\left(\xi_n(\tau) + Q\left(\mho_{w_n,\xi_n}(\tau)\right)\right)d\tau : n \ge 1\right\}, \\ v \in T_0, \\ \chi\{\kappa_1(\vartheta)(w_0 - g(w_n)(0)) : n \ge 1\} \\ + \sum_{k=1}^{k=m} \chi\{\kappa_1(\vartheta - \vartheta_k)I_k(w_n(\vartheta_k^-)) : n \ge 1\} \\ + \chi\{\int_0^\vartheta \kappa_2(\vartheta - \tau)\left(\xi_n(\tau) + Q\left(\mho_{w_n,\xi_n}(\tau)\right)\right)d\tau : n \ge 1\}, \\ v \in T_i, i \in \Pi_m. \end{cases}$$

Now, by (Hg_3), the set $\{g(w_n)(0) : n \ge 1\}$ is relatively compact. Hence, for each $\vartheta \in T$,

$$\chi\{\kappa_1(\vartheta)(\Psi(0) - g(w_n)(0)) : n \ge 1\} = 0. \quad (24)$$

Furthermore, since each I_K is compact for $k \in \Pi_1$, it follows that for each $\vartheta \in T$,

$$\chi\{\kappa_1(\vartheta - \vartheta_k)(I_k(w_n(\vartheta_n^-))) : n \ge 1\} = 0. \quad (25)$$

Next, we estimate the quantity $\chi\left\{\int_0^\vartheta \kappa_2(\vartheta - \tau)\xi_k(\tau)d\tau : k \ge 1\right\}$, where $\vartheta \in T$ is fixed. Note that, from $(H\mathcal{F}_3)$, it holds for $a.e. \vartheta \in T$

$$\begin{aligned} \chi\left\{\xi_k(\vartheta) : k \ge 1\right\} &\le \chi\left\{\xi\left(\vartheta, (w_k)_\vartheta\right) : k \ge 1\right\} \\ &\le \beta(\vartheta) \sup_{\theta \in (-\infty,0]} \chi\{(w_k)_\vartheta(\theta) : k \ge 1\} \\ &= \beta(\vartheta) \sup_{\theta \in (-\infty,0]} \chi\{w_k(\vartheta + \theta) : k \ge 1\} \\ &= \beta(\vartheta) \sup_{\delta \in [0,\vartheta]} \chi\{w_k(\delta) : k \ge 1\} \le \beta(\vartheta)\chi(K(\vartheta)) \end{aligned}$$

$$\le \beta(\vartheta)\chi(K(\vartheta)) = \rho(\vartheta)\ . \quad (26)$$

Note that, $\rho \in L^p(T, R^+)$. According to Bader *et al.* (2001), for every $\varepsilon > 0$, there exists a compact set K_ε, a measurable set $T_\varepsilon \subset T$ with measure less than ε, and a sequence of functions $\{\varsigma_k^\varepsilon\} \subset L^p(T, E)$ such that $\{\varsigma_k^\varepsilon(\tau) : k \ge 1\} \subseteq K_\varepsilon, \tau \in T$ and

$$||\xi_k(\tau) - \varsigma_k^\varepsilon(\tau)|| \le 2\rho(\tau) + \varepsilon, for\ each\ k \ge 1\ and\ each\ \tau \in T - T_\varepsilon. \quad (27)$$

Next, consider the linear continuous operator $G : L^p(T, E) \to C(T, E)$ defined by:

$$G(z)(\vartheta) = \int_0^\vartheta \kappa_2(\vartheta - \tau)z(\tau)d\tau, \forall \vartheta \in T. \quad (28)$$

Combine (26)–(28), for all $\vartheta \in T$ and all $k \ge 1$, we obtain:

$$\begin{aligned} ||G(\xi_k(\vartheta) - G(\varsigma_k^\varepsilon)(\vartheta)|| &\le \eta M_2 \left(\int_0^\vartheta ||\xi_k(\tau) - \varsigma_k^\varepsilon(\tau)||^p\, d\tau\right)^{\frac{1}{p}} \\ &= \eta M_2 [\int_{[0,\vartheta]-T_\varepsilon} ||\xi_k(\tau) - \varsigma_k^\varepsilon(\tau)||^p\, d\tau \\ &\qquad + \int_{[0,\vartheta]\cap T_\varepsilon} ||\xi_k(\tau) - \varsigma_k^\varepsilon(\tau)||^p\, d\tau]^{\frac{1}{p}} \\ &\le \eta M_2 [\int_{[0,\vartheta]-T_\varepsilon} (2\rho(\tau) + \varepsilon)^p\, d\tau \\ &\qquad + \int_{[0,\vartheta]\cap T_\varepsilon} ||\xi_k(\tau) - \varsigma_k^\varepsilon(\tau)||^p\, d\tau]^{\frac{1}{p}} \end{aligned}$$

By taking into account that ε is arbitrary, we get for all $\vartheta \in T$ and all $r \ge 1$

$$||G(\xi_k(\vartheta) - G(\varsigma_k^\varepsilon)(\vartheta)|| \le 2\eta M_2 \left(\int_0^\vartheta (\rho(\tau))^p\, d\tau\right)^{\frac{1}{p}}.$$

Since $\{\varsigma_k^\varepsilon(\tau) : k \ge 1\} \subseteq K_\varepsilon, \chi\{G(\varsigma_k^\varepsilon)(\vartheta) : r \ge 1\} = 0$. Thus, for all $\vartheta \in T$

$$\begin{aligned} &\chi(\left\{\int_0^\vartheta \kappa_2(\vartheta - \tau)\xi_r(\tau)d\tau : r \ge 1\right\}) \\ &= \chi\{G(\xi_r)(\vartheta) : r \ge 1\} \\ &\le \chi\{G(\xi_r)(\vartheta) - G(\varsigma_r^\varepsilon)(\vartheta) : r \ge 1\} \\ &\le \chi\{G(\xi_r)(\vartheta) - G(\varsigma_r^\varepsilon)(\vartheta) : r \ge 1\} \end{aligned}$$

$$\le 2\eta M_2 \left(\int_0^\vartheta (\rho(\tau))^p\, d\tau\right)^{\frac{1}{p}}$$

$$\le 2\eta M_2 \chi_{\mathcal{H}}(Z)||\beta||_{L^p(T,\mathbb{R}^+)}. \quad (29)$$

Next, we estimate the quantity $\chi(\{\int_0^{\vartheta} \kappa_2(\vartheta-\tau)Q(\mho_{w_r,\xi_r}(\tau)d\tau : r \geq 1\})$. From (24), (25), (29) and the fact that $\Delta^{-1}$is linear and bounded, we have:

$$\chi_{L^\infty(T,w)}\{\mho_{w_r,\xi_r}: r \geq 1\}$$
$$\leq N\chi\{w_1 - g(w_r) - \kappa_1(\varpi)(w_0 - g(w_r)) - \sum_{k=1}^{k=m} \kappa_1(\varpi - \vartheta_k)I_k(w_r(\vartheta_k^-)) - \int_0^{\varpi} \kappa_2(\varpi-\tau)\xi_r(\tau)d\tau : r \geq 1\}$$
$$= N\chi\{\int_0^{\varpi} \kappa_2(\varpi-\tau)\xi_r(\tau)d\tau : r \in \mathbb{N}\}$$

(30) $$\leq 2NM_2\eta\chi_{\mathcal{H}}(Z)||\beta||_{L^p(T,\mathbb{R}^+)}.$$

Now, consider the linear continuous operator $\Theta: L^\infty(T,w) \to C(T,E)$, where $\Theta(h)(\vartheta) = \int_0^{\vartheta} \kappa_2(\vartheta-\tau)Q(h(\tau))d\tau$. Then, by (30), we obtain:

$$\chi\left(\left\{\int_0^{\vartheta} \kappa_2(\vartheta-\tau)Q(\mho_{w_r,\xi_r}(\tau)d\tau : r \geq 1\right\}\right) = \chi\{\Theta(\mho_{w_r,\xi_r}): r \geq 1\}$$
$$\leq ||\Theta||\chi_{L^\infty(T,w)}\{\mho_{w_r,\xi_r}: r \geq 1\}$$

(31) $$\leq 2\frac{\varpi^\gamma}{\gamma}N^2M_2^2\eta\chi_{\mathcal{H}}(Z)||\beta||_{L^p(T,\mathbb{R}^+)}.$$

By combining (24), (25), (29) and (31) we obtain:

$$\chi_{PC}(K) = \max_{i=0,1,..,m} \chi_{C(\overline{T_i},E)}(K_{|\overline{T_i}}) \leq \max_{i=0,1,..,m} \max_{\vartheta \in \overline{T_i}} \chi(K(\vartheta))$$
$$= \max_{\vartheta\in T} \chi(K(\vartheta)) \leq \max_{\vartheta\in T} \chi(H(\vartheta)) = \max_{\vartheta\in T} \chi\{y_r(\vartheta): r \geq 1\}$$
$$\leq \chi_{\mathcal{H}}(K)2M_2\eta||\beta||_{L^p(T,\mathbb{R}^+)}\left[1 + N^2\overline{M_\vartheta}\frac{\varpi}{\gamma}\right] < \chi_{PC}(K).$$

This leads to $\chi_{PC}(K)$, and consequently, K is relatively compact.

Step 5. In this step, we demonstrate that, if $\Theta \subseteq B_{n_0}$ is compact, then $R(\Theta)$ is relatively compact. Let $y_n \in R(\Theta), n \geq 1$. Then, there exists $w_n, n \geq 1$, such that $y_n \in R(w_n)$. Hence, there exists $\xi_n \in \tau^p_{\mathcal{F}(.,(w_n)_\vartheta)}$ such that, for $\vartheta \in T$,

$$y_n(\vartheta) = \begin{cases} \kappa_1(\vartheta)(w_0 - g(w)(0)) + \int_0^{\vartheta} \kappa_2(\vartheta-\tau)\left(\xi_n(\tau) + Q\left(\mho_{w_n,\xi_n}(\tau)\right)\right)d\tau, v \in T_0, \\ \kappa_1(\vartheta)(w_0 - g(w)) + \kappa_1(\vartheta-\vartheta_1)I_1(w(\vartheta_1^-)) \\ \quad + \int_0^{\vartheta} \kappa_2(\vartheta-\tau)\left(\xi_n(\tau) + Q\left(\mho_{w_n,\xi_n}(\tau)\right)\right)d\tau, v \in T_1, \\ \quad . \\ \quad . \\ \kappa_1(\vartheta)(w_0 - g(w)(0)) + \sum_{k=1}^{k=m} \kappa_1(\vartheta-\vartheta_k)I_k(w(\vartheta_k^-)) \\ \quad + \int_0^{\vartheta} \kappa_2(\vartheta-\tau)\left(\xi_n(\tau) + Q\left(\mho_{w_n,\xi_n}(\tau)\right)\right)d\tau, v \in T_m. \end{cases}$$

We must show that the set $Z = \{y_n: n \geq 1\}$ is relatively compact in $\mathcal{H}$. As in Step 2, it follows that the set $Z_{|\overline{T_i}}$ is equicontinuous for each $i \in \Pi_0$. In addition, since Θ is compact in $\mathcal{H}$, we have $\sup_{\delta\in[0,\vartheta]} \chi\{w_k(\delta): k \geq 1\} = 0$, and hence, from $(H\mathcal{F}_3)$ we get for $a.e. \vartheta \in T$:

$$\chi\{\xi_k(\vartheta): k \geq 1\} \leq \chi\{\mathcal{F}(\tau,(w_k)_\vartheta): k \geq 1\}$$
$$\leq \beta(\vartheta) \sup_{\theta\in(-\infty,0]} \chi\{(w_k)_\vartheta(\theta): k \geq 1\}$$
$$= \beta(\vartheta) \sup_{\theta\in(-\infty,0]} \chi\{w_k(\vartheta+\theta): k \geq 1\}$$
$$= \beta(\vartheta) \sup_{\delta\in[0,\vartheta]} \chi\{w_k(\delta): k \geq 1\} = 0.$$

Following the arguments in Step 4, we obtains for all $\vartheta \in T$, $\chi\{y_n(\vartheta): n \geq 1\} = 0$, which shows that $R(\Theta)$ is relatively compact.

As a consequence of Steps 1 to 5 and Lemma 2, we conclude that the operator R has a fixed point. Therefore, the problem (1.1) is controllable.

2.2. Controllability result for the system (2):

Consider the following assumptions:

$(H\mathcal{F}_2)^*$ There exists a function $\varphi \in L^p(T,\mathbb{R}^+)$ such that for any $w \in \beta$

(32) $$||\mathcal{F}(\vartheta,w)|| \leq \varphi(\vartheta)(1 + ||w||), \text{ for } a.e. \vartheta \in T.$$

$(Hg_1)^*$ There are $a, d \in (0,\infty)$ with

(33) $$||g(w)(0)||_E \leq a||w||_{\beta_\varpi} + d, \forall w \in \beta_\varpi.$$

$(HI)^*$ For each i$\in \Pi_1$, I_i is continuous and compact and there is $\sigma_i > 0$, such that

(34) $$||I_i(w)|| \leq \sigma_i||w||, w \in E.$$

$(H\Delta)^*$ The linear bounded operator $\Delta: L^p(T,X) \to E$ defined by

$$\Delta(z) = \int_0^{\varpi} \kappa_2(\varpi-\tau)(V_z(\tau))d\tau$$

has a bounded inverse Δ^{-1}.

Theorem 2. Assume that $(H\mathcal{F}_1)$, $(H\mathcal{F}_2)^*$, $(H\mathcal{F}_3)$, $(Hg_1)^*$, (Hg_2), (Hg_3), $(HI)^*$ and $(H\Delta)$ are verified. Then, the system (2) is controllable on T provided that

(35) $$2\overline{M_\vartheta}\eta||\beta||_{L^p(T,\mathbb{R}^+)}\left[1 + \aleph^2\overline{M_\vartheta}\frac{\varpi^\gamma}{\gamma}\right] < 1,$$

and

(36) $$M_1(a+\sigma) + M_2||\varphi||_{L^p(T,\mathbb{R}^+)}\eta + M_2\frac{\varpi^\gamma}{\gamma}\aleph^2\left[a + M_1(a+\sigma) + M_2\eta||\varphi||_{L^p(T,\mathbb{R}^+)}\right] < 1,$$

where η is defined in the statement of Theorem 1, $\sigma = \sum_{i=1}^{i=m} \sigma_i$ and $\aleph > 0$ such that $||\Delta^{-1}|| \leq \aleph$ and $||V|| \leq \aleph$.

Proof. Since the proof is similar to the proof of Theorem1, we will focus on the difference. Using $(H\Delta)$, for any $w \in \mathcal{H}$ and any $\xi \in L^1(T,E)$ with $\xi(\vartheta) \in \mathcal{F}(\vartheta,w_\vartheta), a.e.$, we define the control function $z_{w,\xi} \in L^p(T,w)$ by:

$$z_{w,\xi} = \Delta^{-1}[w_1 - g(w)(0) - \kappa_1(\varpi)(\Psi(0) - g(w)(0)) - \sum_{k=1}^{k=m} \kappa_1(\varpi-\vartheta_k)I_k(w(\vartheta_k^-))$$

(37) $$- \int_0^{\varpi} \kappa_2(\varpi-\tau)\xi(\tau)d\tau].$$

Therefore, we can define a multi valued operator $\mathcal{R}: \mathcal{H} \to 2^{\mathcal{H}}$ as follows: let $w \in \mathcal{H}$. A function $y \in \mathcal{R}(w)$ if and only if

(38) $$y(\vartheta) = \begin{cases} 0, \vartheta \in (-\infty,0], \\ \kappa_1(\vartheta)(\Psi(0) - g(w)(0)) + \int_0^{\vartheta} \kappa_2(\vartheta-\tau)(\xi(\tau) + (V_{z_{w,\xi}})(\tau))d\tau, v \in T_0, \\ \kappa_1(\vartheta)(w_0 - g(w)(0)) + \kappa_1(\vartheta-\vartheta_1)I_1(w(\vartheta_1^-)) \\ \quad . \\ \quad . \\ \quad . \\ \kappa_1(\vartheta)(\Psi(0) - g(w)(0)) + \sum_{k=1}^{k=m} \kappa_1(\vartheta-\vartheta_k)I_k(w(\vartheta_k^-)) \\ \quad + \int_0^{\vartheta} \kappa_2(\vartheta-\tau)\left(\xi(\tau) + (V_{z_{w,\xi}})(\tau)\right)d\tau, v \in T, i = 1,\dots,m, \end{cases}$$

where $\xi \in \tau^1_{\mathcal{F}(.,w_{(.)})}$. As above, we can show that if W is a fixed point for $\mathcal{R}$, then the function $\overline{w}: (-\infty,\varpi] \to E$, defined by:

$$\overline{w}(\vartheta) = \begin{cases} \Psi(\vartheta) - g(w)(\vartheta), \vartheta \in (-\infty,0], \\ w(\vartheta), \vartheta \in [0,\varpi], \end{cases}$$

is a mild solution for Problem (2) and satisfies $w(0) = \Psi(0) - g(w)(0)$ and $w(\varpi) = w_1 - g(w)(0)$. Let $Z_0 = \{w \in \mathcal{H}: ||w|| \leq r\}$, where r is a positive real number such that

(39) $\frac{\varrho}{1-w} < r,$

where

$\varrho = M_1(||w_0|| + d) + M_2||\varphi||_{L^p(T,\mathbb{R}^+)}\eta + M_2\frac{\varpi^\gamma}{\gamma}\aleph^2[||w_1|| + d + M_1(||w_0|| + d)],$

and

$w = M_1(a+\sigma) + M_2||\varphi||_{L^p(T,\mathbb{R}^+)}\eta + M_2\frac{\varpi^\gamma}{\gamma}\aleph^2[a + M_1(a + \delta) + M_2\eta||\varphi||_{L^p(T,\mathbb{R}^+)}].$

Step1. In this step we claim that $\mathcal{R}(Z_0) \subseteq Z_0$. Suppose $w \in Z_0$ and $y \in \mathcal{R}(w)$. Then, there exists $\xi \in \tau^1_{\mathcal{F}(.,w_{(.)})}$ such that y satisfies (38). Since $p > \frac{1}{\gamma}$, the function $\tau \to (\vartheta-\tau)^{\gamma-1}$ belongs to $L^{\frac{p}{p-1}}(T - \mathbb{R}^+)$, hence, using $(H\mathcal{F}_2)^*$ and Hölder's inequality, we get:

$$||\int_0^\vartheta \kappa_2(\vartheta-\tau)\xi(\tau)d\tau|| \le M_2(1+r)\int_0^\vartheta(\vartheta-\tau)^{\gamma-1}\varphi(\tau)d\tau$$

$$\le (1+r)M_2||\varphi||_{L^p(T,\mathbb{R}^+)}\left(\int_0^\vartheta(\vartheta-\tau)^{\frac{(\gamma-1)p}{p-1}}d\tau\right)^{\frac{p-1}{p}}$$

(40) $\le (1+r)M_2\eta||\varphi||_{L^p(T,\mathbb{R}^+),\vartheta\in T}.$

and

$$||\int_0^\vartheta \kappa_2(\vartheta-\tau)(V_{z_w,\xi})(\tau)d\tau|| \le M_2\int_0^\vartheta(\vartheta-\tau)^{\gamma-1}||(V_{z_w,\xi})(\tau)||d\tau$$

(41) $\le M_2||V_{z_w,\xi}||_{L^p(T,E)}\eta.$

Next, according to the definition of $z_{w,\xi}$ and (40), we get:

$$\begin{aligned}||V_{z_w,\xi}||_{L^p(T,E)} &\le ||V||||z_w,\xi||_{L^p(T,X)}\\ &\le ||V||\Delta^{-1}[||w_1|| + ||g(w)(0)|| + ||\kappa_1(\varpi)||(||\Psi(0)|| + ||g(w)(0)||)\\ &\quad + \sum_{k=1}^{k=m}||\kappa_1(\varpi-\vartheta_k)||\,||I_k(w(\vartheta_k^-))||\\ &\quad + \int_0^{\varpi}||\kappa_2(\varpi-\tau)||||\xi(\tau)||d\tau]\\ &\le \aleph^2[||w_1|| + ar + d + M_1(||\Psi(0)|| + ar + d) + M_1 r\,\sigma\\ &\quad + (1+r)M_2\eta||\varphi||_{L^p(T,\mathbb{R}^+)}]\end{aligned}$$

(42) $= \varsigma.$

Combine (41) and (42) to get

(43) $||\int_0^\vartheta \kappa_2(\vartheta-\tau)(V_{z_w,\xi})(\tau)d\tau|| \le M_2\frac{\varpi^\gamma}{\gamma}\varsigma.$

Let $\vartheta \in T_0$. By using (40), (43) and $(H_g)^*$, we get:

(44) $||y(\vartheta)|| \le M_1(||\Psi(0)|| + ar + d) + M_2(1+r)||\varphi||_{L^p(T,\mathbb{R}^+)}\eta + M_2\frac{\varpi^\gamma}{\gamma}\varsigma.$

Likewise, by using $(H_g)^*$ and $(HI)^*$, we get for all $\vartheta \in T_i, i = 1,2,\dots,m,$

$$\begin{aligned}||y(\vartheta)|| &\le M_1(||w_0|| + ar + d + \sigma r)\\ &\quad + M_2(1+r)||\varphi||_{L^p(T,\mathbb{R}^+)}\eta + M_2\frac{\varpi^\gamma}{\gamma}\varsigma\\ &= M_1(||w_0|| + d) + M_2||\varphi||_{L^p(T,\mathbb{R}^+)}\eta + M_2\frac{\varpi^\gamma}{\gamma}\aleph^2[||w_1|| + d + M_1(||w_0|| + d)\\ &\quad + M_2\eta||\varphi||_{L^p(T,\mathbb{R}^+)} + r[M_1(a+\sigma) + M_2||\varphi||_{L^p(T,\mathbb{R}^+)}\eta\end{aligned}$$

(45) $+M_2\frac{\varpi^\gamma}{\gamma}\aleph^2[a + M_1(a+\sigma) + M_2\eta||\varphi||_{L^p(T,\mathbb{R}^+)}].$

In view of (39) and (45), we conclude that $\mathcal{R}(Z_0) \subseteq Z_0$.

Step 2. Let $\kappa = \mathcal{R}(Z_0)$. By following the same arguments from Step 2 and 3 in the proof of Theorem1, one can show that the set $\kappa_{|\overline{T}_i}$ is equicontinuous for each $i \in \Pi_0$, where

$\kappa_{|\overline{T}_i} = \{y^* \in C(\overline{T}_i, E): y^*(\vartheta) = y(\vartheta), \vartheta \in T_i \quad, \quad y^*(\vartheta_i) = y(\vartheta_i^+), y \in \kappa\},$

and the graph of the multivalued function $\mathcal{R}_{|Z_0}: Z_0 \to 2^{Z_0}$ is closed.

Step 3. Let $\mathcal{M} \subseteq Z_0, \mathcal{M} = conv(\{w_0\} \cup \mathcal{R}(\mathcal{M})), \overline{and\ \mathcal{M}} = \bar{C}$ with $C \subseteq \mathcal{M}$ countable. We demonstrate that $\mathcal{M}$ is relatively compact. Because $\mathcal{M}$ is equicontinuous on each $\overline{T}_i, i \in \Pi_0$, we only need to show that $\mathcal{M}(\vartheta)$ is relatively compact in E. From the countability of C and $C \subseteq \mathcal{M} = conv(\{w_0\} \cup \mathcal{R}(\mathcal{M}))$, we can find a countable set $H = \{y_n: n \ge 1\} \subseteq \mathcal{R}(\mathcal{M})$ with $C \subseteq conv(\{w_0\} \cup H)$. Therefore,

$\chi(\mathcal{M}(\vartheta)) \le \chi(\bar{C}(\vartheta)) \le \chi(H(\vartheta)) = \chi\{y_n(\vartheta): n \ge 1\}, \vartheta \in T.$

In the same manner as in the fourth step of the proof of Theorem 1, we can demonstrate that

$\chi\{y_n(\vartheta): n \ge 1\} \le \chi_{\mathcal{H}}(Z)2\,\overline{M_\vartheta}\eta||\beta||_{L^p(T,\mathbb{R}^+)}[1 + \aleph^2\overline{M_\vartheta}\frac{\varpi^\gamma}{\gamma}],$

Thus, by (35), $\chi_{PC}(\mathcal{M}) = 0$.

Step 4. $\mathcal{R}$ maps compact sets into relatively compact sets.

This can be proven by following the same arguments as in Step 5 of the proof of Theorem 1.

Finally, by applying Lemma 2, the proof is complete.

Corollary 1. If we replaced $(H\mathcal{F}_2)^*$ with:

$(H\mathcal{F}_2)^{**}$ There is a $\varphi \in L^p(T,\mathbb{R}^+)$ such that for any $w \in E$ $||\mathcal{F}(\vartheta,w)|| \le \varphi(\vartheta),\ a.e. \vartheta \in T.$

then, the condition (36) becomes

(46) $M_1(a+\sigma) + M_2\frac{\varpi^\gamma}{\gamma}\aleph^2[a + M_1(a+\sigma)] < 1.$

Proof. Let $r > 0$ and suppose it satisfies

(47) $\frac{\xi}{1-M_1(a+\sigma)+M_2\frac{\varpi^\gamma}{\gamma}\aleph^2[a+M_1(a+\sigma)]} < r,$

where

$$\xi = M_1(||\Psi(0)|| + d) + M_2\eta||\varphi||_{L^p(T,\mathbb{R}^+)}$$

(48) $+ M_2\frac{\varpi^\gamma}{\gamma}\aleph^2[||w_1|| + d + M_1(||\Psi(0)|| + d) + M_2\eta||\varphi||_{L^p(T,\mathbb{R}^+)}].$

We just need to check $\mathcal{R}(B_0) \subseteq B_0$, where $B_0 = \{w \in \mathcal{H}: ||w|| \le r\}$. Let $w \in B_0$ and $y \in R(w)$. Then, there is $\xi \in \tau^p_{\mathcal{F}(.,w(.))}$ such that y satisfies (38). As in Step 1 of the proof of Theorem 2, we obtain the estimate:

$$\begin{aligned}||y(\vartheta)|| &\le M_1(||\Psi(0)|| + ar + d + \sigma r) + M_2||\varphi||_{L^p(T,\mathbb{R}^+)}\eta\\ &\quad + M_2\frac{\varpi^\gamma}{\gamma}\varsigma\\ &= M_1(||\Psi(0)|| + ar + d + \sigma r) + M_2||\varphi||_{L^p(T,\mathbb{R}^+)}\eta\\ &\quad + M_2\frac{\varpi^\gamma}{\gamma}\aleph^2[||w_1|| + ar + d + M_1(||\Psi(0)|| + ar + d)\\ &\quad + M_1 r\sigma + M_2\eta||\varphi||_{L^p(T,\mathbb{R}^+)}].\\ &= M_1(||\Psi(0)|| + d) + M_2||\varphi||_{L^p(T,\mathbb{R}^+)}\eta\\ &\quad + M_2\frac{\varpi^\gamma}{\gamma}\aleph^2[||w_1|| + d + M_1(||\Psi(0)|| + d) + M_2\eta||\varphi||_{L^p(T,\mathbb{R}^+)}].\end{aligned}$$

(49) $+r\left[M_1(a+\sigma) + M_2\frac{\varpi^\gamma}{\gamma}\aleph^2(a + M_1(a+\sigma))\right];\ \vartheta \in T.$

It follows from (46)–(49) that $\mathcal{R}(\varpi_0) \subseteq B_0$.

Remark 1. The controllability of (2) can be achieved by adopting the assumptions and arguments used in Theorem 1. The same applies to the controllability of (1).

Example

Assume that $\varrho: (-\infty, 0] \to (-\infty, 0]$ is a continuous function with $L = \int_{-\infty}^0 \varrho(\tau)d\tau < \infty$, and let B_ϱ be the vector space of all functions

$w: (-\infty, 0] \to E$ which is bounded and measurable on $[-r, 0]$ for each $r > 0$, and satisfy $\int_{-\infty}^{0} \varrho(\tau) \sup_{\vartheta \in [\tau,0]} ||w(\vartheta)|| \, d\tau < \infty$. It is known that, B_ϱ is a phase space that fulfils all assumptions of Definition 1, with the norm given by $||w||_{B_\varrho} = \int_{-\infty}^{0} \varrho(\tau) \sup_{\vartheta \in [\tau,0]} ||w(\vartheta)|| \, d\tau$ (Karthikeyan *et al.*, 2021).

Let $\Omega = \{\tau = (\tau_1, \tau_2): \tau_1^2 + \tau_2^2 \leq 1\}$, and $E = w = L^2(\Omega)$. Define an operator $A: D(A) \subseteq E \to E$ by

(50) $$A(u) := \Delta u - u,$$

with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. It is known that (Ren *et al.*, 2019) A is a sectorial operator. Let $T = [0,1], m = 2, 0 = \vartheta_0 < \vartheta_1 = \frac{1}{3} < \vartheta_2 = \frac{2}{3} < \vartheta_3 = 1$, Z be a non-empty, compact and convex subset of $E, \upsilon = sup_{z \in Z} ||z||$. Consider $\mathcal{F}: T \times B_\varrho \longrightarrow P_{ck}(E)$ defined by:

(51) $$\mathcal{F}(\delta, \psi) = \{z \in E: z(\tau) = \frac{e^{-r\vartheta}\sqrt{\tau_1^2+\tau_2^2}||\psi||}{\delta(1+||\psi||)} Z, \tau = (\tau_1, \tau_2)\},$$

where $r > 1$ then,

$$||\mathcal{F}(\delta, \psi)|| = \sup_{z \in \mathcal{F}(\delta,\psi)} ||z||_E = \sup_{z \in \mathcal{F}(\vartheta,\psi)} [\int_\Omega ||z(\tau)||^2 \, d\tau]^{\frac{1}{2}}$$
$$= \frac{e^{-r\vartheta}||\psi||}{(1+||\psi||)} [\int_\Omega (\tau_1^2 + \tau_2^2) \, d\tau]^{\frac{1}{2}}$$

(52) $$\leq e^{-r\vartheta}||\psi|| < e^{-r\vartheta}(||\psi|| + 1).$$

In addition, let $\vartheta \in T, \psi_1, \psi_2 \in B_\varrho$ and $z_1 \in \mathcal{F}(\vartheta, \psi_1)$. Then,

$$z_1 = \frac{e^{-r\vartheta}\sqrt{\tau_1^2+\tau_2^2} \sup_{\theta \in (-\infty,0]} ||\psi_1(\theta)||w}{\upsilon}, w \in Z.$$

Put $$z_2 = \frac{e^{-r\vartheta}\sqrt{\tau_1^2+\tau_2^2} \sup_{\theta \in (-\infty,0]} ||\psi_2(\vartheta)||w}{\upsilon}, w \in Z$$.

Obviously, $z_2 \in \mathcal{F}(\vartheta, \psi_2)$ and

$$||z_1 - z_2|| \leq e^{-r\vartheta}[\sup_{\theta \in (-\infty,0]} ||\psi_1(\theta)|| - \sup_{\theta \in (-\infty,0]} ||\psi_2(\theta)||][\int_\Omega |\tau| d\tau]^{\frac{1}{2}}$$
$$= e^{-r\vartheta} \sup_{\theta \in (-\infty,0]} (||\psi_1(\theta)|| - ||\psi_2(\theta)||)$$
$$\leq e^{-r\vartheta} \sup_{\theta \in (-\infty,0]} ||\psi_1(\theta) - \psi_2(\theta)||,$$

which yields

(53) $$h(\mathcal{F}(\vartheta, \Phi_1), \mathcal{F}(\vartheta, \Phi_2)) \leq e^{-r\vartheta} \sup_{\theta \in (-\infty,0]} ||\psi_1(\theta) - \psi_2(\theta)||, \forall \vartheta \in T, \Phi_1, \Phi_2 \in B_\varrho.$$

It follows from (52) that, for any bounded subset, Ω , of B_ϱ one has

$$\chi(\mathcal{F}(\vartheta, \Omega) \leq e^{-r\vartheta} \sup_{\theta \in (-\infty,0]} \chi\{\psi(\theta): \psi \in \Omega\}.$$

Then, assumptions $(H\mathcal{F}_1), (H\mathcal{F}_2)^*$ and $(H\mathcal{F}_3)$ are satisfied. Let $g: B_\varpi \longrightarrow B$ be defined by

(54) $$g(w)(\vartheta) = \lambda \Upsilon(w(\vartheta)); \ \vartheta \in (-\infty, 0], w \in B_\varpi,$$

where $\lambda > 0$ and $\Upsilon: E \to E$ is a linear, bounded and compact operator. Notice that $||g(w)(0)|| \leq \lambda||\Upsilon|| ||w(0)|| \leq ||\Upsilon|| ||w||_{B_\varpi}$. Therefore, $(Hg_1)^*$ is verified with $a = \lambda||\Upsilon||$ and $d = 0$. Let $w_n \to w$ in B_ϖ. Then, $w_n(0) \longrightarrow w(0)$ in E, and hence $g(w_n)(0) \to g(w)(0)$. Moreover, if D is a bounded subset in B_ϖ, then, from the compactness of ϑ , the set $\{g(w)(0): w \in D\}$ is relatively compact in E . So, (Hg_2) and (Hg_3) are satisfied. Let $I_i: E \to E \ (i = 1, 2)$ be defined by

(55) $$I_i(w) = \sigma_i \Upsilon(w),$$

where σ_i are positive real numbers. Next, suppose $V: L^p(T, w) \to L^p(T, E)$ is a bounded linear operator such that the operator $\Delta: L^p(T, w) \to E$ defined by

(56) $$\Delta(z) = \int_0^\varpi \kappa_2(\varpi - \tau)(V_z)(\tau))d\tau,$$

has a bounded inverse $\Delta^{-1}: E \to L^p(T, w)/Ker(\Delta)$. Let $\aleph > 0$ with $||\Delta^{-1}|| \leq \aleph$ and $||V|| \leq \aleph$. By choosing $r, \lambda, \vartheta, \sigma_1, \sigma_2$ such that the inequalities (35) and (36) are satisfied, we can apply Theorem 2. Therefore, the system (2) is controllable, where $A, \mathcal{F}, g, I_i, V$ and Δ are given by (51) and (53)– (56).

3. Discussion and Conclusion

In recent years, there have been many contributions concerning various kinds of controllability of different types of differential equations and inclusions involving fractional derivatives. Some of these works have considered problems with finite or infinite delay, infinite delay and impulsive effects. Others have treated problems generated by semigroups of operators, sectorial operators and almost sectorial operators. However, to the best of our knowledge, the controllability of Caputo fractional differential inclusions generated by sectorial operators and in the presence of nonlocal conditions, impulses and infinite delay has not yet been treated.

In this paper, we fill this gap and prove the exact controllability of two fractional differential inclusions generated by sectorial operators in infinite-dimensional spaces and with impulses, infinite delay and nonlocal conditions.

We did not assume, similar to Alsaroria and Ghadle (2022), that the families of operators $\{\kappa_1(\vartheta): \vartheta > 0\}$ and $\{\kappa_2(\vartheta): \vartheta > 0\}$, which are generated by A, are compact, and this increases the importance of this work.

Our technique was based on the properties of phase spaces, fractional sectorial operators, multi-valued functions, the Hausdorff measure of noncompactness and a fixed-point theorem for multi-valued functions.

Since fractional calculus has many applications in medicine, energy and other fields of science, this work contributes to these applications.

Many directions for future work are possible. Indeed, our technique can be used to:

a. Generalise the results in Alsheekhhussain and Ibrahim (2021) and Kumar *et al.* (2022) to the case when the nonlinear term is a multi-valued function.
b. Extend the obtained results in Abbas (2020), Almarri and Elshenhab (2022), Varun *et al.* (2022) and Mohan *et al.* (2024), in the presence of impulses, nonlocal conditions and infinite delay, and when the linear term is a fractional sectorial operator and the nonlinear term is a multi-valued function.
c. Extend the work in both Almarri and Elshenhab (2022) and Karthikeyan *et al.* (2021) when the delay is infinite.
d. Study the controllability of the considered problem in Raja *et al.* (2025).

Data Availability Statement

The data that support the findings of this study are available on request from the corresponding author.

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Conflicts of Interest

No conflicts of interest exist.

Biographies

Feryal Abdullah Al Adsani

Department of Mathematics, College of Science, King Faisal University, Al Ahsa, Saudi Arabia, 0096666505934144, faladsani@kfu.edu.sa

Al Adsani is a graduate of King Faisal University (KFU) and currently serves as an assistant professor in Saudi Arabia. In 2011, she was appointed Head of the Mawhiba Program at KFU. She also chaired the Qiyas Testing Committee affiliated with the National Center for Assessment and Evaluation at Al-Kifah Schools during 2014 and 2015. Throughout her career, she has published several research articles, held various leadership roles within the university, and actively contributed to numerous committees at the departmental, college, and university levels.

ORCID ID: 0000-0003-3549-5587

Ahmed Gamal Ibrahim

Independent Researcher, Faculty of Science, Cairo University, Cairo, Egypt, 00201019429069, agamal2000@yahoo.com

Prof. Ibrahim, an Egyptian mathematician with a Ph.D. from Cairo University, specializes in functional analysis and differential inclusions. With over 45 years of academic experience, he has authored 72 publications, supervised 15 theses, and co-authored a widely used Functional Analysis textbook. He served as a professor at KFU and visiting professor at MIU. An active reviewer for international journals, he regularly participates in global conferences. His research focuses on functional analysis, set-valued functions, and integer/fractional-order differential inclusions.

ORCID ID: 0000-0001-7416-5973

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