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An Efficient Hybrid Conjugate Gradient Algorithm for Solving Intuitionistic Fuzzy Nonlinear Equations

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Abstract

This paper presents an iterative algorithm for solving intuitionistic fuzzy nonlinear equations (IFNEs). The proposed method is based on the classical conjugate gradient (CG) search direction. An interesting feature of the new algorithm is that it considers problems based on the special triangular intuitionistic fuzzy number. For this purpose, intuitionistic fuzzy quantities are transformed into membership and non-membership parametric forms, and a line search procedure is employed to compute the step length. Preliminary results from numerical experiments are presented to demonstrate the performance of the method. It is observed that the proposed hybrid CG method is highly effective and promising.

KEYWORDS

Hybrid CG, intuitionistic fuzzy nonlinear equation, parametric form, step length, inexact line search

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1. Introduction

Consider the following nonlinear equation:

$$F(x) = 0. \quad (1)$$

The numerical solution of (1) plays an important role in engineering, mathematics, astrophysics, economics, and natural sciences. Interestingly, numerical approaches for these types of problems consider cases where the coefficients are fuzzy numbers rather than crisp numbers (Abbasbandy and Asady, 2004; Waziri and Moyi, 2016. Sulaiman *et al.*, 2018; Umar *et al.*, 2018; Sulaiman *et al.*, 2020). The fuzzy number system is a branch of the intuitionistic fuzzy set that has recently gained much attention because of its numerous real-life applications. Atanassov (1999) introduced the intuitionistic fuzzy set as a general form of the fuzzy set theory presented by Zadeh (1996). The intuitionistic fuzzy set contains the degree of membership and non-membership to the set. Despite the numerous practical applications of the Intuitionistic fuzzy set, only a few studies have been conducted to find the numerical solution to the problem. The standard analytic approach of Buckley and Qu (1990, 1991) considers the quadratic and linear case of fuzzy equations only with the initial point selected near the solution point. The method of Buckley and Qu (1990, 1991) is limited and not suitable for solving nonlinear equations of the form:

(1) $ax^3 + bx^2 + cx - d = e$
(2) $px^2 + qcos(x) = r$
(3) $cx^2 + d = e$

Where $a, b, c, d, e, p, q, and\ r$ are intuitionistic fuzzy numbers. Numerous numerical algorithms have been developed for solving fuzzy nonlinear equations to overcome this drawback (Yang *et al.*, 2008; Abbasbandy and Asady, 2004; Kajani *et al.*, 2005; Sulaiman *et al.*, 2016; Mohammed *et al.*, 2020; Umar *et al.*, 2020a; Sulaiman *et al.*, 2021, 2022b). However, only a few works of literature have investigated the performance of new iterative methods for solving equations whose coefficients are intuitionistic. Some of these studies are presented by Amma *et al.* (2016), who applied Euler and Taylor algorithms for solving intuitionistic fuzzy differential equations, and Biswas *et al.* (2016), who employed the Adomian decomposition scheme to solve differential equations with fuzzy coefficient, containing the linear differential operator. Additionally, Ettoussi *et al.* (2015) studied the successive approximation approach for solving fuzzy differential equations of intuitionistic nature. It can be observed that the above literature focused on intuitionistic differential equations, where the coefficients were fuzzy numbers. Perhaps, only a few works of literature are available on algorithms for solving intuitionistic fuzzy nonlinear problems. One of the available studies was presented by Keyanpour and Akbarian (2014), who applied the mid-point Newton algorithm to solve the IFNE.

Most of the methods discussed above are variants of Newton's method, whose Jacobian or approximate Jacobian matrix is computed and stored during the iteration process, making them costly and time-consuming. Therefore, this study considers a gradient-based approach derived from combining two CG coefficients. The CG algorithms are unconstrained optimization processes characterized by their simplicity, nice convergence properties, low memory requirements, and less computational cost in the iteration process (Malik *et al.*, 2020). For more details on hybrid conjugate methods, refer to (Touati-Ahmed and Storey, 1990; Gilbert and Nocedal, 1992; Dai and Yuan, 2001; Andrei, 2008; Liu and Li, 2014; Yakubu *et al.*, 2020; Malik *et al.*, 2021; Sulaiman and Mamat, 2020a; Sulaiman *et al.*, 2022a). This paper defines a new hybrid CG algorithm using an inexact line search procedure to solve intuitionistic fuzzy nonlinear problems.

The remaining part of this study is structured as follows: Section 2 presents the preliminary results and basic definition of terms, followed by an overview of the proposed hybrid CG algorithm in Section 3. The hybrid CG algorithm for IFNEs is discussed in Section

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4, and implementation on benchmark problems is demonstrated in Section 5. Finally, the conclusion is presented in Section 6.

2. Preliminaries

Various definitions of intuitionistic fuzzy numbers are presented in the following sections (Dubois, 1980; Goetschel and Voxman, 1986; Shaw and Roy, 2012).

2.1. Definition 1 (Goetschel and Voxman, 1986; Shaw and Roy, 2012)

A fuzzy number of the real line R is a fuzzy set A with membership functions $\mu_A: R \to [0,1]$, satisfying

a) A is normal, i.e., there exists an element x_0 such that $\mu_A(x_0) = 1$
b) A is fuzzy convex for membership function $\mu_A(x)$, i.e., $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq min(\mu_A(x_1), \mu_A(x_2)), \forall\, x_1, x_2 \in R, \forall \lambda \in [0,1]$,
c) A is fuzzy concave for the non-membership function $\nu_A(x)$, i.e., $\nu_A(\lambda x_1 + (1-\lambda)x_2) \leq max(\nu_A(x_1), \nu_A(x_2))\, \forall\, x_1, x_2 \in R, \lambda \in [0,1]$,
d) μ_A is upper-semi continuous,
e) $Supp\ A$ is bounded.

2.2. Definition 2 (Dubois, 1980; Shaw and Roy, 2012)

An intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \vartheta_A(x)\rangle | x \in R\}$ of the real line is called an intuitionistic fuzzy number if

a) A is intuitionistic fuzzy normal; then, there exist at least two points $x_0, x_1 \in X$ satisfying $\mu_A(x_0) = 1$ and $\nu_A(x_1) = 1$,
b) A is intuitionistic fuzzy convex, i.e., its membership function μ is fuzzy concave, and its non-membership function ν is fuzzy convex,
c) μ_A is upper semicontinuous and ν_A is lower semicontinuous,
d) $Supp\ A = \{x \in X | \nu_A(x) < 1\}$ is bounded.

2.3. Definition 3 (Goetschel and Voxman, 1986; Shaw and Roy, 2012)

An intuitionistic fuzzy set $A \in E$ is defined as an object of form

$$A = \{\langle x, \mu_A(x), \vartheta_A(x)\rangle | x \in E\},$$

where

$$\vartheta_A: E \to [0,1]$$

and

$$\mu_A: E \to [0,1]$$

defines the degree of membership and non-membership of the elements $x \in E$ respectively, and for every $x \in E$:

$$0 \leq \mu_A(x) + \vartheta_A(x) \leq 1.$$

2.4. Definition 4 (Goetschel and Voxman, 1986; Shaw and Roy, 2012)

The parameterized form of the intuitionistic fuzzy number n is a pair, $n = ([\underline{n}, \overline{n}], [\underline{\underline{n}}, \overline{\overline{n}}])$, of function $\underline{n}, \overline{n}, \underline{\underline{n}}, \overline{\overline{n}}$, satisfying that

1. $\underline{n}(\alpha)$ is a bounded monotonic increasing left continuous function.
2. $\overline{n}(\alpha)$ is a bounded monotonic decreasing left continuous function.
3. $\underline{\underline{n}}(\alpha)$ is a bounded monotonic increasing left continuous function.
4. the $\overline{\overline{n}}(\alpha)$ is a bounded monotonic decreasing left continuous function.
5. $\underline{n}(\alpha) \leq \overline{n}(\alpha), \underline{\underline{n}}(\alpha) \leq \overline{\overline{n}}(\alpha), 0 \leq \alpha \leq 1.$

2.5. Definition 5 (Goetschel and Voxman, 1986; Shaw and Roy, 2012)

A triangular intuitionistic fuzzy number $\langle u, v\rangle$ is an intuitionistic fuzzy set in R with membership function u and membership function v defined by

$$v(x) = \begin{cases} \frac{a_2 - x}{a_2 - a_1^1}, & a_1' \leq x \leq a_2 \\ \frac{x - a_2}{a_3^1 - a_2}, & a_2 \leq x \leq a_3' \\ 1, & otherwise \end{cases} \text{ and } u(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\ \frac{a_3 - x}{a_3 - a_2}, & a_2 \leq x \leq a_3 \\ 0, & otherwise \end{cases}$$

where $u(x), v(x) \leq 0.5$ for $u(x) = v(x), \forall x \in R$ and $a_1' \leq a_1 \leq a_2 \leq a_3 \leq a_3'$.

The triangular intuitionistic fuzzy number is represented by $\langle u, v\rangle = \langle a_1, a_2, a_3; a_1', a_2, a_3'\rangle$, where the parameterized form is given as

$$\overline{u}(\alpha) = a_3 - \alpha(a_3 - a_2), \quad \underline{u}(\alpha) = a_1 + \alpha(a_2 - a_1)$$
$$\overline{\overline{v}}(\alpha) = a_3' - \alpha(a_3' - a_2), \quad \underline{\underline{v}}(\alpha) = a_1' + \alpha(a_2 - a_1')$$

3. Hybrid CG Algorithm Formulation for Unconstrained Optimization

The CG algorithms with formulas given as,

$$\beta_k^{FR} = \frac{g_k^T g_k}{\|g_{k-1}\|^2} \quad \text{and} \quad \beta_k^{DY} = \frac{g_k^T g_k}{(g_k - g_{k-1})^T d_{k-1}}, \tag{2}$$

are presented by Fletcher and Reeves (FR) (1964) and Dai-Yuan (1999). These methods possess strong convergence conditions despite their poor computational performance due to the jamming phenomenon (Al-Baali, 1985). Similarly, the CG methods of Polak and Ribière (1969), Polyak (1969), and Hestenes-Stiefel (1952) with formulas given as,

$$\beta_k^{PR} = \frac{g_k^T (g_k - g_{k-1})}{\|g_{k-1}\|^2} \quad \text{and} \quad \beta_k^{HS} = \frac{g_k^T (g_k - g_{k-1})}{(g_k - g_{k-1})^T d_{k-1}}, \tag{3}$$

are known to possess efficient numerical performance but may not necessarily converge to the solution point. The idea of hybridization uses the efficient numerical performance of algorithms defined in Eq. (3) and the nice convergence properties from methods given in Eq. (2) to determine a new and efficient variant of the CG method. The first hybrid CG algorithm was presented by Touti-Ahmed and Storey (1990).

Recently, Umar *et al.* (2020b) presented a nonlinear CG method based on the FR algorithm, with the formula for β_k defined as

$$\beta_k^{UM} = \frac{\|g_k\|^2}{d_{k-1}^T\left(d_{k-1} - \frac{\|g_k\|}{\|g_{k-1}\|} g_k - g_k\right)}. \tag{4}$$

The authors discussed this method's convergence analysis and descent condition under exact minimization conditions.

Based on the above modification, this study defined a novel hybrid CG parameter denoted by β_k^{UMFR} with UMFR denoting the authors, Umar, Mustafa, Fletcher, and Reeves, with the formula given as

$$\beta_k^{UMFR} = min\left\{0, max\left\{\frac{\|g_k\|^2}{d_{k-1}^T\left(d_{k-1} - \frac{\|g_k\|}{\|g_{k-1}\|} g_k - g_k\right)}, \frac{g_k^T g_k}{\|g_{k-1}\|^2}\right\}\right\}. \tag{5}$$

It is obvious that (5) is a convex combination of

$$\beta_k^{UM} = \frac{\|g_k\|^2}{d_{k-1}^T\left(d_{k-1} - \frac{\|g_k\|}{\|g_{k-1}\|} g_k - g_k\right)} \quad \text{and} \quad \beta_k^{FR} = \frac{g_k^T g_k}{\|g_{k-1}\|^2}.$$

It can be observed that both β_k^{UM} and β_k^{FR} have the same numerator. Refer to Al-Baali (1985) for FR convergence proof under an inexact line search.

4. Hybrid CG Algorithm for IFNE

Consider IFNE, $F(x) = 0$, whose parameterized form is given as

$$\mu_A(x) = \begin{cases} \underline{F}(\underline{x}, \overline{x}, r) = 0, \\ \overline{F}(\overline{x}, \underline{x}, r) = 0, \end{cases} \quad \nu_A(x) = \begin{cases} \underline{\underline{F}}(\underline{\underline{x}}, \overline{\overline{x}}, r) = 0 \\ \overline{\overline{F}}(\overline{\overline{x}}, \underline{\underline{x}}, r) = 0 \end{cases} \quad \forall r \in [0,1]. \tag{6}$$

This section aims to obtain the solution to Eq. (6) via the proposed CG algorithm. By defining the function $G_r: R^2 \to R$ as

$$G_r(\underline{x}, \overline{x}; \underline{\underline{x}}, \overline{\overline{x}}) = [\mu_A(x) + \nu_A(x)]^2, \ \forall r \in [0,1] \tag{7}$$

whose gradient is given as $\nabla G_r(x)$ at $x(r) = (\underline{x}(r), \overline{x}(r))$ for membership function and $x(r) = \left(\underline{\underline{x}}(r), \overline{\overline{x}}(r)\right)$ for non-membership function—note that

$$\nabla G_r(x) = \left(\frac{\partial G_r}{\partial \underline{x}}, \frac{\partial G_r}{\partial \overline{x}}, \frac{\partial G_r}{\partial \underline{\underline{x}}}, \frac{\partial G_r}{\partial \overline{\overline{x}}}\right). \tag{8}$$

Based on the definition of $G_r(\underline{x}, \overline{x}; \underline{\underline{x}}, \overline{\overline{x}}, r)$ in Eq. (7), we can transform Eq. (6) into the following unconstrained optimization problem:

$$\min_{x\in E} G_r(x).$$

Further, this section aims to define the efficient hybrid CG algorithm for $x_k(r) = x_{k-1}(r) + \alpha_{k-1}d_k$, where α_{k-1} is the step size calculated via the following inexact line search process:

$$\omega_1\|\alpha g_k\|^2 \le \|g(x_k+\alpha p_k)\|^2 - \|g_k\|^2 \le -\omega_2\|\alpha g_k\|^2 \qquad (9)$$

and the search direction d_k updated using

$$d_k = \begin{cases} -\nabla G_r(x_k) & if\ k=0 \\ -\nabla G_r(x_k) + \beta_k d_{k-1} & if\ k\ge 1 \end{cases}. \qquad (10)$$

Here, β_k^{UMFR} possess the descent property. Based on the description above, it is obvious that for $\forall r\in[0,1]$, the solution $(\underline{x}^*, \overline{x}_*, \underline{\underline{x}}^*, \bar{\bar{x}}_*)$ satisfying $G_r(\underline{x}^*, \overline{x}_*, \underline{\underline{x}}^*, \bar{\bar{x}}_*) = 0$ is the same for Eq. (6) and contrariwise.

If we consider the starting guess for $\underline{x}(0) \le \overline{x}(1) \le \overline{x}(0)$, and $\underline{\underline{x}}(0) \le \overline{x}(1) \le \bar{\bar{x}}(0)$, the fuzzy number

$$x_0 = (\underline{x}(0) \le \overline{x}(1) \le \overline{x}(0)) \quad \text{and}$$

$$x_0 = \underline{\underline{x}}(0) \le \overline{x}(1) \le \bar{\bar{x}}(0), \qquad (11)$$

can be chosen with the parameterized form defined for the membership function, given as

$$\overline{x}(r) = \overline{x}(0) + \left(\underline{x}(1) - \overline{x}(0)\right) r \text{ and } \underline{x}(r) = \underline{x}(0) + \left(\underline{x}(1) - \underline{x}(0)\right) r,$$

And the non-membership function as

$$\bar{\bar{x}}(r) = \bar{\bar{x}}(0) + \left(\underline{\underline{x}}(1) - \bar{\bar{x}}(0)\right) r \text{ and } \underline{\underline{x}}(r) = \underline{\underline{x}}(0) + \left(\underline{\underline{x}}(1) - \underline{\underline{x}}(0)\right) r.$$

Next, the algorithm for implementing the proposed hybrid CG method is presented as follows:

Algorithm 1: Hybrid CG algorithm for IFNE

Step 1: Transforming the defined IFNE into a parameterized form

and obtaining the initial points by solving for r = 0 and r = 1.

Step 2: Evaluating G_r at $(\underline{x}(0), \overline{x}(0))$ and $(\underline{\underline{x}}(0), \bar{\bar{x}}(0))$ to obtain $\nabla G_r(x_0)$

Step 3: if $\|\nabla G_r(x_0)\| \le 0$, terminate. Else set $d_0 = -\nabla G_r(x_0)$

Step 4: if $k=0$, evaluate α_k based on Eq. (9)

Step 5: Set $x_{k+1}(r) = x_k(r) + \alpha_k d_k$,

Step 6: Calculate β_k by Eq. (5) and d_k using Eq. (10)

Step 7: Set $k = k+1$, then continue the process from stage Eq. (1) to Eq. (6) until convergence is achieved.

5. Numerical Experiments

This section presents the experimental results of some intuitionistic fuzzy nonlinear problems to validate the performance of the CG algorithm. All computations were carried out on MATLAB 2015a version using a double precision operating system. The computational outcome for the proposed method is presented in Table 1 and Table 2, and the graphical representation to illustrate the efficiency of the new method is given in Figure 1 and Figure 2.

Problem 1: Consider the IFNE

$$(3,4,5; 4,5,6)x^2 + (1,2,3; 2,3,3)x = (1,2,3; 2,3,3).$$

Without loss of generality, let x be positive. The parameterized form of the problem containing the membership and non-membership elements is obtained as follows:

$$(3+r)\underline{x}^2(r) + (1+r)\underline{x} = (1+r)$$

$$(5-r)\overline{x}^2(r) + (3-r)\overline{x} = (3-r)$$

$$(4+r)\underline{\underline{x}}^2(r) + (2+r)\underline{\underline{x}} = (2+r)$$

$$(6-r)\bar{\bar{x}}^2(r) + (3-r)\bar{\bar{x}} = (3-r)$$

Further, this section aims to obtain the initial points by solving the parameterized equations for $r=0$ and $r=1$.

For $r = 0$

$$(3)\underline{x}^2(0) + (1)\underline{x} = (1)$$

$$(5)\overline{x}^2(0) + (3)\overline{x} = (3)$$

$$(4)\underline{\underline{x}}^2(0) + (2)\underline{\underline{x}} = (2)$$

$$(6)\bar{\bar{x}}^2(0) + (3)\bar{\bar{x}} = (3)$$

and $r = 1$

$$(4)\underline{x}^2(r) + (2)x = (2)$$

$$(4)\overline{x}^2(r) + (2)x = (2)$$

$$(5)\underline{\underline{x}}^2(r) + (3)\underline{\underline{x}} = (3)$$

$$(5)\bar{\bar{x}}^2(r) + (2)x = (2).$$

Thus,

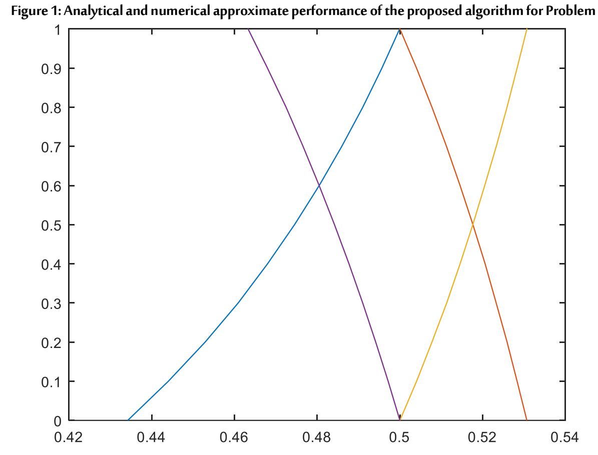
$x(0) = (0.4342, 0.5307, 0.5000, 0.5000, 0.5307, 0.4633)$, i.e., $\underline{x}(0), \overline{x}(1) = \underline{\underline{x}}(0), \bar{\bar{x}}(1)$. The above initial point values for $r = 0\ and\ r = 1$ are near the solution point. To demonstrate the suitability of the CG algorithm in solving Problem 1, we consider the initial guess as $x_0 = (0.4, 0.5, 0.4, 0.5)$. Using the proposed method, we obtained the solution after seven iterations with the maximum error set as $error < 10^{-5}$. Further, $\forall r\in[0,1]$, we obtained the analytical solution presented in Table 1.

Table 1: Analytical solution of problem 1 $\forall t \in [0,1]$

t	$\underline{x}(t)$	$\overline{x}(t)$	$\underline{\underline{x}}(t)$	$\bar{\bar{x}}(t)$
0	0.434219896359385	0.530693573746126	0.500003739508119	0.500000000000000
0.1	0.444041048589348	0.528331777730026	0.504036296993986	0.497166694382989
0.2	0.452927830082607	0.525916081833066	0.507749247230498	0.494159768146865
0.3	0.460920743931371	0.523275850113425	0.511318674995232	0.491014408037575
0.4	0.468013982290471	0.520604282626075	0.514556537648764	0.487701430649126
0.5	0.474540425975309	0.517658660646051	0.517631154086397	0.484193909474672
0.6	0.480507361018480	0.514544587633762	0.520506669359078	0.480492047747159
0.7	0.485959604292724	0.511265670844620	0.523299707991484	0.476596741246770
0.8	0.491033925593414	0.507757346447644	0.525897346098635	0.472494729596972
0.9	0.495661894511351	0.504022123939370	0.528331529645268	0.467995850520970
1.0	0.499948196306483	0.500000000000000	0.530698905586760	0.463305891865044

Also, based on the analytical results presented above, we presented the performance profile in Figure 1.

Figure 1: Analytical and numerical approximate performance of the proposed algorithm for Problem 1



Problem 2: Consider the intuitionistic fuzzy problem,

$$(3,4,5; 2,3,4)x^2 + (1,2,3; 1,2,2)x = (1,2,3; 1,2,2).$$

Without loss of generality, let x be positive. The parameterized form of the problem containing the membership and non-membership elements is obtained as follows:

$$(3+r)\underline{x}^2(r) + (1+r)\underline{x}(r) = (1+r)$$

$$(5-r)\overline{x}^2(r) + (3-r)\overline{x}\,(r) = (3-r)$$

$$(2+r)\underline{x}^2(r) + (1+r)\underline{x}(r) = (1+r)$$

$$(4-r)\overline{x}^2(r)+(2-r)\overline{x}\,(r)=(2-r)$$

Further, this section aims to obtain the initial points by solving the parameterized equations for $r = 0$ and $r = 1$.

For $r = 0$

$$3\underline{x}^2(0)+\underline{x}(0)=1$$
$$5\overline{x}^2(0)+3\overline{x}\,(0)=3$$
$$2\underline{x}^2(0)+\underline{x}(0)=1$$
$$4\overline{x}^2(0)+2\overline{x}\,(0)=2$$

and $r = 1$

$$4\underline{x}^2(1)+2\underline{x}(1)=2$$
$$4\overline{x}^2(1)+2\overline{x}\,(1)=2$$
$$3\underline{x}^2(1)+2\underline{x}(1)=2$$
$$3\overline{x}^2(1)+\overline{x}\,(1)=1$$

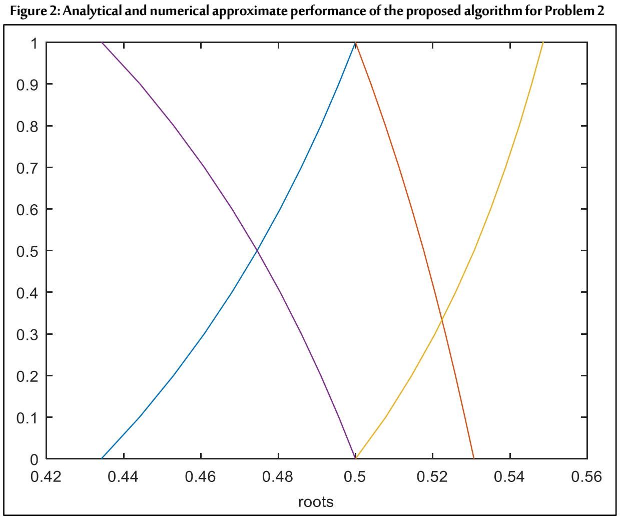
Thus, $x(0) = (0.4342,0.5307,0.5000,0.5000,0.5486,0.4343)$, i.e., $\underline{x}(0),\overline{x}(1)=\underline{\underline{x}}(0),\overline{\overline{x}}(1)$. The above initial values for $r = 0\ and\ r = 1$ are near the solution point. To demonstrate the suitability of the CG algorithm in solving Problem 2, we consider the initial guess as $x_0 = (0.6,0.7,0.6,0.7)$. Using the proposed hybrid method, we obtained the solution after six iterations with a maximum error considered as $\epsilon < 10^{-5}$. . Further, $\forall r \in [0,1]$, we obtain the analytical solution presented in Table 2.

Table 2: Analytical solution of problem 2 $\forall t \in [0,1]$

t	$\underline{x}(t)$	$x\,(t)$	$\underline{x}(t)$	$x\,(t)$
0	0.434214513085421	0.530680183565861	0.500015256890315	0.500000000000000
0.1	0.444101427122724	0.528316209677886	0.507849478767587	0.495676876969925
0.2	0.452906674264892	0.525880510701867	0.514559904813682	0.491020149978603
0.3	0.460849129789527	0.523295260625865	0.520604118811853	0.485976696504712
0.4	0.468026686794300	0.520548723393640	0.525922897491075	0.480506147583310
0.5	0.474548224684487	0.517671662041831	0.530747214602492	0.474546376978471
0.6	0.480506112521205	0.514576067521989	0.534999439036038	0.468024276584467
0.7	0.485976437720592	0.511282104178793	0.538861157625467	0.460852328387588
0.8	0.491029508356102	0.507751920744983	0.542410791795435	0.452932061468196
0.9	0.495663822986187	0.504010160544257	0.545616257876011	0.444229133498361
1.0	0.499991444854872	0.500000000000000	0.548584202556072	0.434304740999126

Also, based on the analytical results presented above, we plot the performance profile in Figure 2.

Figure 2: Analytical and numerical approximate performance of the proposed algorithm for Problem 2



6. Conclusion

This study examined the performance of a hybrid CG algorithm on intuitionistic fuzzy nonlinear problems. The process began by parameterizing the intuitionistic fuzzy nonlinear problems. Based on the parametric form, the initial point was obtained by solving for $r = 0$ and $r = 1$. Finally, the problems were solved by applying the proposed hybrid CG method. Throughout this study, we considered the inexact line search procedure for the experiment. Preliminary results from the numerical experimentation are encouraging because we considered the initial points further away from those obtained. The outcome shows that the new algorithm is efficient and promising.

Biographies

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